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DILOGARITHM AND HIGHER $\mathscr{L}$-INVARIANTS FOR $\mathrm{GL}_3(\mathbf{Q}_p)$

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ABSTRACT. The primary purpose of this paper is to clarify the relation between previous results in [Ann. Sci. Éc. Norm. Supér. 44 (2011), pp. 43–145], [Amer. J. Math. 141 (2019), pp. 661–703], and [Camb. J. Math. 8 (2020), p. 775–951] via the construction of some interesting locally analytic representations. Let E be a sufficiently large finite extension of $\mathbf{Q}_p$ and ρ_p be a p-adic semi-stable representation $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \to \mathrm{GL}_3(E)$ such that the associated Weil–Deligne representation $\mathrm{WD}(\rho_p)$ has rank two monodromy and the associated Hodge filtration is non-critical. A computation of extensions of rank one (φ, Γ)-modules shows that the Hodge filtration of ρ_p depends on three invariants in E. We construct a family of locally analytic representations $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of $\mathrm{GL}_3(\mathbf{Q}_p)$ depending on three invariants $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$, such that each representation in the family contains the locally algebraic representation $\mathrm{Alg} \otimes \mathrm{Steinberg}$ determined by $\mathrm{WD}(\rho_p)$ (via classical local Langlands correspondence for $\mathrm{GL}_3(\mathbf{Q}_p)$) and the Hodge–Tate weights of ρ_p. When ρ_p comes from an automorphic representation π of a unitary group over $\mathbf{Q}$ which is compact at infinity, we show (under some technical assumption) that there is a unique locally analytic representation in the above family that occurs as a subrepresentation of the Hecke eigenspace (associated with π) in the completed cohomology. We note that [Amer. J. Math. 141 (2019), pp. 611–703] constructs a family of locally analytic representations depending on four invariants (cf. (4) in that publication) and proves that there is a unique representation in this family that embeds into the Hecke eigenspace above. We prove that if a representation Π in Breuil's family embeds into the Hecke eigenspace above, the embedding of Π extends uniquely to an embedding of a $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ into the Hecke eigenspace, for certain $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ uniquely determined by Π. This gives a purely representation theoretical necessary condition for Π to embed into completed cohomology. Moreover, certain natural subquotients of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ give an explicit complex of locally analytic representations that realizes the derived object $\Sigma(\lambda, \underline{\mathscr{L}})$ in (1.14) of [Ann. Sci. Éc. Norm. Supér. 44 (2011), pp. 43–145]. Consequently, the locally analytic representation $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ gives a relation between the higher $\mathscr{L}$-invariants studied in [Amer. J. Math. 141 (2019), pp. 611–703] as well as the work of Breuil and Ding and the p-adic dilogarithm function which appears in the construction of $\Sigma(\lambda, \underline{\mathscr{L}})$ in [Ann. Sci. Éc. Norm. Supér. 44 (2011), pp. 43–145].

CONTENTS

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1. Introduction

Let p be a prime number and F an imaginary quadratic extension of $\mathbf{Q}$ such that p splits in F. We fix a unitary group G over $\mathbf{Q}$ which splits over F and such that $G(\mathbf{R})$ is compact. Then to each finite extension E of $\mathbf{Q}_p$ and to each prime-to-p level U^p in $G(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$, one can associate the Banach space of p-adic automorphic forms $\widehat{S}(U^p,E)$. One can also associate with U^p a set of finite places $D(U^p)$ of $\mathbf{Q}$ and a Hecke algebra $\mathbf{T}(U^p)$ which is the polynomial algebra freely generated by Hecke operators at places of F lying above $D(U^p)$. In particular, the commutative algebra $\mathbf{T}(U^p)$ acts on $\widehat{S}(U^p,E)$ and commutes with the action of $G(\mathbf{Q}_p)\cong \mathrm{GL}_n(\mathbf{Q}_p)$ coming from translations on $G(\mathbf{A}_{\mathbf{Q}}^{\infty})$.

If $\rho\colon \mathrm{Gal}(\overline{F}/F)\to \mathrm{GL}_n(E)$ is a continuous irreducible representation, one considers the associated Hecke eigenspace $\widehat{S}(U^p,E)[\mathfrak{m}_\rho]$, which is a continuous admissible representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ over E, or its locally $\mathbf{Q}_p$-analytic vectors $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$, which is an admissible locally $\mathbf{Q}_p$-analytic representation of $\mathrm{GL}_n(\mathbf{Q}_p)$. We fix w_p to be a place of F above p. The philosophy of p-adic local Langlands correspondence predicts that $\widehat{S}(U^p,E)[\mathfrak{m}_\rho]$ (and its subspace $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ as well) determines and depends only on $\rho_p \overset{\mathrm{def}}{=} \rho|_{\mathrm{Gal}(\overline{F_{w_p}}/F_{w_p})}$. The case $n=2$ is well-known essentially due to various results in [Col10] and [Eme]. The case $n\geq 3$ is much more difficult and only a few partial results are known. We are particularly interested in the case when the subspace of locally algebraic vectors $\widehat{S}(U^p,E)^{\mathrm{alg}}[\mathfrak{m}_\rho]\subsetneq \widehat{S}(U^p,E)[\mathfrak{m}_\rho]$ is non-zero, which implies that ρ_p is potentially semi-stable. Certain cases when $n=3$ and ρ_p is semi-stable and non-crystalline have been studied in [Bre17] and [BD20]. We are going to continue their work and obtain some interesting relation between results in [Bre17], [BD20] and previous results in [Schr11] which involve the p-adic dilogarithm function.

1.1. Construction of a family of representations.

We consider a weight $\lambda\in X(T)_+$ of the diagonal split torus $T\subseteq \mathrm{GL}_3$ which is dominant with respect to the upper-triangular Borel subgroup. Given two locally analytic representations V_1, V_2 of $\mathrm{GL}_3(\mathbf{Q}_p)$, we use the notation $V_1 \text{ --- } V_2$ (resp. the notation $V_1 \text{ - - } V_2$) for a locally analytic representation corresponding to a non-zero (resp. possibly zero) element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}(V_2,\ V_1)$. If we consider two elements in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}(V_2,\ V_1)$ that differ from each other by a non-zero scalar, then their corresponding representations are naturally isomorphic. In Section 2.3, we will introduce the generalized analytic Steinberg representations (of weight λ) $\mathrm{St}_3^{\mathrm{an}}(\lambda)$, $v_{P_1}^{\mathrm{an}}(\lambda)$, $v_{P_2}^{\mathrm{an}}(\lambda)$, $\overline{L}(\lambda)$ and various irreducible locally analytic representations $C^*_{w',w}$ of $\mathrm{GL}_3(\mathbf{Q}_p)$, for certain choices of $*\in\{\varnothing,1,2\}$ and elements w,w' in the Weyl group of GL_3.

Theorem 1.1 (Proposition 6.2, Proposition 6.8, Proposition 6.12, (6.42))**.** *For each choice of $\lambda\in X(T)_+$ and $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$, there exists a locally analytic* *representation* $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ *of* $\mathrm{GL}_3(\mathbf{Q}_p)$ *of the form:*

$$
\begin{tikzcd}[column sep=small]
 & & C_{s_1,s_1} \arrow[r, dash] & \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty} \arrow[dr, dash] \arrow[dddr, dash] & \\
 & v_{P_1}^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[rrr, dash] & & & \overline{L}(\lambda) \\
\mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & & & \\
 & v_{P_2}^{\mathrm{an}}(\lambda) \arrow[dr, dash] \arrow[rrr, dash] & & & \overline{L}(\lambda) \\
 & & C_{s_2,s_2} \arrow[r, dash] & \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty} \arrow[ur, dash] \arrow[uuur, dash] &
\end{tikzcd} \tag{1.1}
$$

Moreover, different choices of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ *give non-isomorphic representations.*

We also construct a locally analytic representation $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \supsetneq \Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form

$$
\begin{tikzcd}[column sep=small]
 & & & C^1_{s_2s_1,s_2s_1} \arrow[dr, dash] & \\
 & & C_{s_1,s_1} \arrow[ur, dash] \arrow[r, dash] & \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty} \arrow[r, dash] \arrow[dr, dash] \arrow[dddr, dash] & C^2_{s_1,s_1s_2} \\
 & v_{P_1}^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[rrr, dash] & & & \overline{L}(\lambda) \\
\mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & & & \\
 & v_{P_2}^{\mathrm{an}}(\lambda) \arrow[dr, dash] \arrow[rrr, dash] & & & \overline{L}(\lambda) \\
 & & C_{s_2,s_2} \arrow[r, dash] \arrow[dr, dash] & \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty} \arrow[r, dash] \arrow[ur, dash] \arrow[uuur, dash] & C^2_{s_2,s_2s_1} \\
 & & & C^1_{s_1s_2,s_1s_2} \arrow[ur, dash] &
\end{tikzcd}
$$

whose isomorphism class is uniquely determined by that of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$. The following is our main result on local-global compatibility.

Theorem 1.2 (Theorem 7.1)**.** *Assume that* $p \geq 5$ *and* $n = 3$*. Assume moreover that*

(i) ρ *is unramified at all finite places of* F *above* $D(U^p)$*;*
(ii) $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{alg}} \neq 0$*;*
(iii) ρ_p *is semi-stable with Hodge–Tate weights* $\{k_1 > k_2 > k_3\}$ *such that* $N^2 \neq 0$*;*
(iv) ρ_p *is non-critical in the sense of Remark 6.1.4 of* [Bre17]*;*
(v) *only one automorphic representation contributes to* $\widehat{S}(U^p, E)^{\mathrm{alg}}[\mathfrak{m}_\rho]$.

Then there exists a unique choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ *such that* $\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ *contains (copies of) the locally analytic representation*

$$\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (k_1 - 2, k_2 - 1, k_3)$*, and* $\alpha \in E^\times$ *is determined by the Weil–Deligne representation* $\mathrm{WD}(\rho_p)$ *associated with* ρ_p*. Moreover, we have*

$$
\begin{aligned}
&\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \\
&\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^{\infty} \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right).
\end{aligned} \tag{1.2}
$$

The assumptions of our Theorem 1.2 are the same as that of Theorem 1.3 of [Bre17]. Here we do not attempt to obtain any explicit relation between $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ and ρ_p, which is similar in flavor to Theorem 1.3 of [Bre17]. The improvement of our Theorem 1.2 upon Theorem 1.3 of [Bre17] will be explained in Section 1.3. It is worth mentioning that, under further technical assumptions that ρ_p is ordinary with consecutive Hodge–Tate weights and has an irreducible mod p reduction, one can combine our Theorem 1.2 with Theorem 7.52 of [BD20] and conclude that the isomorphism class of $\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and that of ρ_p determine each other.

Remark 1.3. It is possible to construct a locally analytic representation $\Sigma^{\max}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ of $\mathrm{GL}_3(\mathbf{Q}_p)$ containing $\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ which is characterized by the fact that it is maximal (for inclusion) among the locally analytic representations V satisfying the following conditions:

(i) $\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V)=V^{\mathrm{alg}}=\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$;
(ii) each constituent of V is a subquotient of a locally analytic principal series;
(iii) $\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$ is a Jordan–Hölder factor of V with multiplicity one,

where V^{alg} is the subspace of locally algebraic vectors in V. Moreover, an immediate generalization of the arguments in the proof of Theorem 1.2 (and thus of Theorem 1.1 of [Bre17]) shows that

$$\begin{aligned}(1.3)\quad &\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\max}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right)\\ &\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right).\end{aligned}$$

One can also show that

$$\Sigma^{\max}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)/\overline{L}(\lambda)\otimes_E\mathrm{St}_3$$

is independent of the choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$. However, the full construction of $\Sigma^{\max}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ is very lengthy and technical, and thus we decided not to put it here.

1.2. **Derived object and p-adic dilogarithm.** We consider the bounded derived category

$$\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

associated with the abelian category $\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}$ of abstract modules over the algebra $D(\mathrm{GL}_3(\mathbf{Q}_p),E)$ consisting of locally $\mathbf{Q}_p$-analytic distributions on $\mathrm{GL}_3(\mathbf{Q}_p)$ (cf. Section 4 of [ST03] for the definition of the algebra of distributions). Schraen constructs an object

$$\Sigma(\lambda,\underline{\mathscr{L}})'\in\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

in Definition 5.19 of [Schr11], and this construction crucially involves the p-adic dilogarithm function. However, it was not clear in [Schr11] whether there exists an explicit complex $[C_\bullet]$ of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ whose strong dual realizes $\Sigma(\lambda,\underline{\mathscr{L}})'$. Upon minor difference between the notation of [Schr11] and ours, we show that

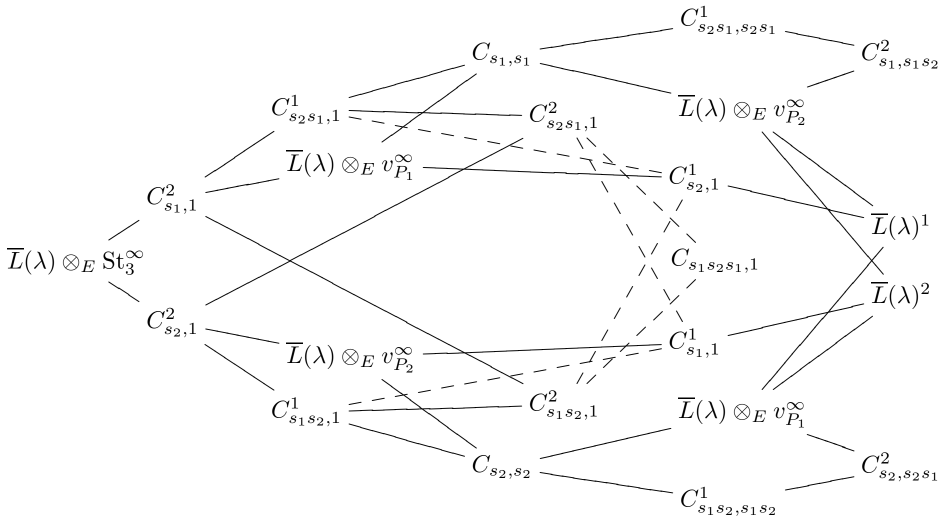
Theorem 1.4 (Theorem 6.15, (2.23))**.** *There exists an explicit complex $[C_\bullet]$ of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ such that the object*

$$\mathcal{D}'\in\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

associated with $\left[C'_{-\bullet}\right]$ satisfies

$$\mathcal{D}'\cong\Sigma(\lambda,\underline{\mathscr{L}})'\in\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right).$$

1.3. **Higher $\mathscr{L}$-invariants for** $\mathrm{GL}_3(\mathbf{Q}_p)$**.** It follows from (6.43) and (6.44) that $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ can be described explicitly by the following picture:



Consequently, $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ contains a unique subrepresentation of the form

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & C^1_{s_2s_1,1} \ar[dr, dash] & & C^1_{s_2s_1,s_2s_1} \ar[dr, dash] & \\
 & C^2_{s_1,1} \ar[ur, dash] \ar[dr, dash] & & C_{s_1,s_1} \ar[ur, dash] \ar[dr, dash] & & C^2_{s_1,s_1s_2} \\
 & & \overline{L}(\lambda)\otimes_E v_{P_1}^\infty \ar[ur, dash] & & \overline{L}(\lambda)\otimes_E v_{P_2}^\infty \ar[ur, dash] & \\
\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \ar[uur, dash] \ar[ddr, dash] & & & & & \\
 & & \overline{L}(\lambda)\otimes_E v_{P_2}^\infty \ar[dr, dash] & & \overline{L}(\lambda)\otimes_E v_{P_1}^\infty \ar[dr, dash] & \\
 & C^2_{s_2,1} \ar[ur, dash] \ar[dr, dash] & & C_{s_2,s_2} \ar[ur, dash] \ar[dr, dash] & & C^2_{s_2,s_2s_1} \\
 & & C^1_{s_1s_2,1} \ar[ur, dash] & & C^1_{s_1s_2,s_1s_2} \ar[ur, dash] &
\end{tikzcd}$$

which is denoted by

$$\begin{tikzcd}[row sep=tiny]
 & \Pi^1(\underline{k},\underline{D}) \\
\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \ar[ur, dash] \ar[dr, dash] & \\
 & \Pi^2(\underline{k},\underline{D})
\end{tikzcd} \tag{1.4}$$

in Theorem 1.1 of [Bre17]. We write Π for an arbitrary representation of the form (1.4). It follows from Theorem 1.2 of [Bre17] that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\Pi^i(\underline{k},\underline{D}),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 3$$

for each $i = 1, 2$. Therefore all possible choices of Π form a family that depends on four invariants in E. However, a computation of extensions of rank one (φ,Γ)-modules suggests that ρ_p depends on three invariants in E. As a result, Theorem 1.1 of [Bre17] predicts that the existence of U^p and ρ as well as an embedding $\Pi \hookrightarrow \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$, should cut out a subfamily of Π that depends on three invariants. Motivated by Breuil's prediction, we show the following

Theorem 1.5 (Corollary 7.5)**.** *If there exists U^p and ρ such that Π embeds into $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$, then there exists $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3 \in E$ such that Π embeds into*

$$\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3).$$

Moreover, the isomorphism class of Π *and that of* $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ *where* Π *embeds, uniquely determine each other.*

1.4. **Sketch of content.** The overall goal of the sections before Section 7 is the construction and study of the locally analytic representations $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$. In particular, the content of this paper from Section 2 to Section 6 is purely locally analytic representation theoretical.

In Section 2, we recall various well-known facts around locally analytic representations of p-adic analytic groups, with more focus on $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\mathrm{GL}_3(\mathbf{Q}_p)$. In Section 2.3, we fix our notation for various locally analytic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\mathrm{GL}_3(\mathbf{Q}_p)$, including the notation for some irreducible admissible locally analytic representations for $\mathrm{GL}_3(\mathbf{Q}_p)$ that will be frequently used in the rest of the article. In Section 2.2, we recall a standard spectral sequence (cf. Lemma 2.1) which will be frequently used in later computation of Ext-groups. In Section 2.4, we fix a branch of the p-adic logarithm function, recall a branch of the p-adic dilogarithm function from Section 5.3 of [Schr11] and interpret it as an element of a certain $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p)}$-group following (5.57) of [Schr11]. Using the fixed branch of the p-adic logarithm function, we define a locally analytic representation $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ of $\mathrm{GL}_3(\mathbf{Q}_p)$ that depends on two invariants $\mathscr{L}_1,\mathscr{L}_2\in E$ (cf. the paragraph before (2.23)).

In Section 3, we prove a crucial fact (Proposition 3.5) on the non-existence of a locally analytic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ of a certain specific form, which can be interpreted as the vanishing of a certain $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}$-group. The proof of Proposition 3.5 uses arguments involving infinitesimal characters of locally analytic representations.

In Section 4, we systematically present a list of computational results, grouped into various Propositions and Lemmas. There exists a standard spectral sequence (cf. Lemma 2.1) to compute certain $\mathrm{Ext}_{\mathrm{GL}_3(\mathbf{Q}_p)}$-groups using results on $N_P(\mathbf{Q}_p)$-homology of admissible locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$, where N_P is the unipotent radical of a maximal parabolic subgroup $P\subsetneq \mathrm{GL}_3$. Consequently, our computation in Section 4 makes extensive use of results on $N_P(\mathbf{Q}_p)$-homology, most notably Théorème 4.10 of [Schr11] (a classical Theorem by Kostant) as well as Section 5.2 and 5.3 of [Bre17] (based on the lists between (4.117) and (4.134) of [Schr11]). The readers may skip Section 4 during a first reading. While reading Section 5 and 6, the reader may check the lists in Section 4 whenever necessary.

In Section 5, we prove various technical results on Ext-groups that will be directly used in the construction and study of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ (which appears in Section 6). On the one hand, we prove in Proposition 5.4 the non-existence of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ of certain specific forms, using Proposition 3.5 as a crucial input. On the other hand, we compute or estimate the dimension of various $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}$ and $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p)}$ in Lemma 5.3, 5.5, 5.7, 5.8 and 5.9. Technically speaking, the information on dimensions of these Ext-groups will be crucial for us to manipulate various long exact sequences in Section 6.

Section 6 is the heart of this paper, where we construct and study the representation $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and its variant. In Section 6.1, we finish the construction of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ (cf. Proposition 6.8 and the paragraph before (6.28)), and then prove a technical result (cf. Proposition 6.10) which will be crucial in the proof of Theorem 7.1. In Section 6.2, we further clarify the structure of various subrepresentations of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and obtain an explicit description of extensions inside $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ (cf. (6.42) and (6.43)). In order to clarify the relation between our $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and various representations constructed in [Bre17] (cf. the proof of Theorem 7.1 for details), we also consider a slightly bigger representation $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3) \supsetneq \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$. In Section 6.3, we obtain as byproduct an explicit complex (cf. Theorem 6.15) of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ that realizes the derived object $\Sigma(\lambda,\underline{\mathscr{L}})'$ constructed in [Schr11].

In Section 7, we prove Theorem 7.1 by combining Proposition 6.10 with the technique (recalled or reformulated in Proposition 7.2, 7.3 and 7.4) from the proof of Théorème 6.2.1 of [Bre17]. At the end, we give a purely representation theoretical criterion for a representation of the form (1.4) to embed into the completed cohomology (cf. Corollary 7.5).

2. Preliminary

2.1. **Locally analytic representations.** In this section, we recall some background on the theory of locally analytic representations of p-adic analytic groups.

We fix a locally $\mathbf{Q}_p$-analytic group H and denote the algebra of locally $\mathbf{Q}_p$-analytic distributions with coefficients in E on H by $\mathcal{D}(H,E)$, which is defined as the strong dual of the locally convex E-vector space $C^{\mathrm{an}}(H,E)$ consisting of locally $\mathbf{Q}_p$-analytic functions on H (cf. Section 4 of [ST03]). We use the notation $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ (resp. $\mathrm{Rep}^{\infty}_{H,E}$) for the category of admissible locally $\mathbf{Q}_p$-analytic representations of H (resp. admissible smooth representations of H) with coefficients in E. It follows from Theorem 6.3 of [ST03] that taking strong dual induces a fully faithful contravariant functor from $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ to the abelian category $\mathrm{Mod}_{\mathcal{D}(H,E)}$ of abstract modules over $\mathcal{D}(H,E)$. The E-vector space $\mathrm{Ext}^i_{\mathcal{D}(H,E)}(M_1,M_2)$ is well-defined for any two objects $M_1,M_2\in\mathrm{Mod}_{\mathcal{D}(H,E)}$, and we define

$$\mathrm{Ext}^i_H(\Pi_1,\Pi_2)\overset{\mathrm{def}}{=}\mathrm{Ext}^i_{\mathcal{D}(H,E)}(\Pi_2',\Pi_1')$$

for any two objects $\Pi_1,\Pi_2\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ where $\cdot'$ is the notation for strong dual. We also define the cohomology of an object $M\in\mathrm{Mod}_{\mathcal{D}(H,E)}$ by

$$H^i(H,M)\overset{\mathrm{def}}{=}\mathrm{Ext}^i_{\mathcal{D}(H,E)}(1_H',M)$$

where 1_H is the trivial representation of H. If H_1 is a closed locally $\mathbf{Q}_p$-analytic normal subgroup of H, then H/H_1 is also a locally $\mathbf{Q}_p$-analytic group. It follows from the fact

$$D(H,E)\otimes_{D(H_1,E)}E\cong D(H/H_1,E)$$

(cf. Section 5.1 of [Bre17]) that $H^i(H_1,M)$ admits a structure of $\mathcal{D}(H/H_1,E)$-module for each $M\in\mathrm{Mod}_{\mathcal{D}(H,E)}$. For each $\Pi\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$, if there exists an object $H_i(H_1,\Pi)\in\mathrm{Rep}^{\mathrm{la}}_{H/H_1,E}$ such that

$$H_i(H_1,\Pi)'\cong H^i(H_1,\Pi'),$$

we call $H_i(H_1,\Pi)$ the H_1-homology of Π. Note that $H_i(H_1,\Pi)$, if exists, is well-defined up to isomorphism due to Theorem 6.2 of [ST03]. Throughout this paper, whenever we use the notation $H_i(H_1,\Pi)$ for certain normal subgroup $H_1\subseteq H$ and certain $\Pi\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$, we implicitly mean that $H_i(H_1,\Pi)$ exists as an object of $\mathrm{Rep}^{\mathrm{la}}_{H/H_1,E}$. We fix a subgroup Z inside the center of H. Then the algebra $\mathcal{D}(Z,E)$, consisting of locally $\mathbf{Q}_p$-analytic distribution on Z with coefficients in E, is naturally contained in the center of $\mathcal{D}(H,E)$. For each locally $\mathbf{Q}_p$-analytic E-character χ of Z, we define $\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$ as the abelian subcategory of $\mathrm{Mod}_{\mathcal{D}(H,E)}$ consisting of all the objects on which $\mathcal{D}(Z,E)$ acts by χ'. We write $\mathrm{Ext}^i_{\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}}(-,-)$ for the usual Ext-groups inside the abelian category $\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$. Then we define

$$\mathrm{Ext}^i_{H,\chi}(\Pi_1,\Pi_2)\overset{\mathrm{def}}{=}\mathrm{Ext}^i_{\mathcal{D}(H,E),\chi'}(\Pi_2',\Pi_1')$$

for any two objects $\Pi_1,\Pi_2\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ such that $\Pi_1',\Pi_2'\in\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$. In particular, if Z is the center of H and acts on $\Pi\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ via the character χ, then $\Pi'\in\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$, and we usually say that Π admits a central character χ.

Assume now that H is the set of $\mathbf{Q}_p$-points of a split reductive group over $\mathbf{Q}_p$. We fix a maximal torus and a Borel subgroup $T\subseteq B\subseteq H$ and call a parabolic subgroup $P\subseteq H$ *standard* if it contains B. We write $\overline{P}\subseteq H$ for the opposite parabolic subgroup with $L=P\cap\overline{P}$ the *standard Levi subgroup* of P. We also write N (resp. $\overline{N}$) for the unipotent radical of P (resp. of $\overline{P}$), and use the notation $\mathfrak{h}$, $\mathfrak{p}$, $\mathfrak{n}\ldots$ for the E-Lie algebras associated with $H\times_{\mathbf{Q}_p}E$, $P\times_{\mathbf{Q}_p}E$, $N\times_{\mathbf{Q}_p}E\ldots.$ We consider the category $\mathcal{O}$ together with its subcategory $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ for each parabolic subgroup $P\subseteq H$ (cf. Section 9.3 of [Hum08] or [OS15]). For each parabolic subgroup $P\subseteq H$ with Levi quotient L, we have the Orlik–Strauch functor

$$\mathcal{F}^H_P:\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}\times\mathrm{Rep}^{\infty}_{L,E}\to\mathrm{Rep}^{\mathrm{la}}_{H,E}.$$

The nice properties of $\mathcal{F}^H_P$ are summarized in the main theorem of [OS15].

2.2. **Formal properties.** In this section, we summarize some general formal properties of locally analytic representations of p-adic reductive groups. We fix a split p-adic reductive group H throughout this section.

We consider a parabolic subgroup $P\subseteq H$ with unipotent radical N and Levi quotient L.

Lemma 2.1. *We consider $\Pi_1\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ and $\Pi_2\in\mathrm{Rep}^{\mathrm{la}}_{L,E}$ such that*

(i) *$H_k(N,\ \Pi_1)\in\mathrm{Rep}^{\mathrm{la}}_{L,E}$ exists for each $k\geq 0$;*
(ii) *the (*FIN*) condition in Section 6 of* [ST05] *holds for Π_2.*

Then there exists a spectral sequence

$$\mathrm{Ext}^j_{L,*}\left(H_k(N,\ \Pi_1),\ \Pi_2\right)\Rightarrow\mathrm{Ext}^{j+k}_{H,*}\left(\Pi_1,\ \mathrm{Ind}^H_P\left(\Pi_2\right)^{\mathrm{an}}\right)$$

for each $*\in\{\varnothing,\chi\}$ where χ is a locally analytic character of the center of H. In particular, we have an isomorphism

$$\mathrm{Hom}_{L,*}\left(H_0(N,\ \Pi_1),\ \Pi_2\right)\xrightarrow{\sim}\mathrm{Hom}_{H,*}\left(\Pi_1,\ \mathrm{Ind}^H_P\left(\Pi_2\right)^{\mathrm{an}}\right)$$

and a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{L,*}\left(H_0(N,\ \Pi_1),\ \Pi_2\right)&\hookrightarrow\mathrm{Ext}^1_{H,*}\left(\Pi_1,\ \mathrm{Ind}^H_P\left(\Pi_2\right)^{\mathrm{an}}\right)\\&\to\mathrm{Hom}_{L,*}\left(H_1(N,\ \Pi_1),\ \Pi_2\right)\to\mathrm{Ext}^2_{L,*}\left(H_0(N,\ \Pi_1),\ \Pi_2\right)\end{aligned}$$

for each $*\in\{\varnothing,\chi\}$.

Proof. This follows directly from (44) and (45) of [Bre17] as well as our definition of $\mathrm{Ext}^k_{H,*}$, $\mathrm{Ext}^k_{L,*}$ and H_k in Section 2.1 for each $k\geq 0$. $\square$

We fix a finite length locally analytic representation $V \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ equipped with an increasing filtration of subrepresentations $\{\mathrm{Fil}_k V\}_{0\leq k\leq m}$ such that

$\mathrm{Fil}_0(V) = 0$, $\mathrm{Fil}_m(V) = V$ and $\mathrm{gr}_{k+1}V \stackrel{\mathrm{def}}{=} \mathrm{Fil}_{k+1}V/\mathrm{Fil}_k V \neq 0$ for all $0 \leq k \leq m-1$.

Note that the assumption above automatically implies that

$$\ell(V) \geq m$$

where $\ell(V)$ is the length of V.

Proposition 2.2. *Assume that V_1 is another object of $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ and χ is a locally analytic character of the center of H.*

(i) *If $\mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{gr}_k V) = 0$ for each $1 \leq k \leq m$, then we have*

$$\mathrm{Ext}^1_{H,\chi}(V_1,\ V) = 0.$$

(ii) *If there exists $1 \leq k_0 \leq m$ such that $\mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{gr}_k V) = 0$ for each $1 \leq k \neq k_0 \leq m$ and $\dim_E \mathrm{Ext}^1_{H,\chi}\left(V_1,\ \mathrm{gr}_{k_0} V\right) = 1$, then we have*

$$\dim_E \mathrm{Ext}^1_{H,\chi}(V_1,\ V) \leq 1;$$

if moreover $\mathrm{Ext}^2_{H,\chi}(V_1,\ \mathrm{gr}_k V) = 0$ for each $1 \leq k \leq k_0-1$ and $\mathrm{Hom}_{H,\chi}(V_1,\ \mathrm{gr}_k V) = 0$ for each $k_0+1 \leq k \leq m$, then we have

$$\dim_E \mathrm{Ext}^1_{H,\chi}(V_1, V) = 1.$$

Proof. For each $1 \leq k \leq m-1$, the short exact sequence $\mathrm{Fil}_k V \hookrightarrow \mathrm{Fil}_{k+1}V \twoheadrightarrow \mathrm{gr}_{k+1}V$ induces a long exact sequence

$$\mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_k V) \to \mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k+1} V) \to \mathrm{Ext}^1_{H,\chi}\left(V_1,\ \mathrm{gr}_{k+1} V\right)$$

which implies

$$\dim_E \mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k+1} V) \leq \dim_E \mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_k V) + \dim_E \mathrm{Ext}^1_{H,\chi}\left(V_1,\ \mathrm{gr}_{k+1} V\right).$$

Therefore we finish the proof of part (i) and the first claim of part (ii) by induction on k and the fact that $\mathrm{gr}_1 V = \mathrm{Fil}_1 V$.

Now we prove the second claim of part (ii). The same method as in the proof of part (i) shows that

$$\mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0-1} V) = \mathrm{Ext}^2_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0-1} V) = 0 \tag{2.1}$$

and

$$\mathrm{Ext}^1_{H,\chi}(V_1,\ V/\mathrm{Fil}_{k_0} V) = \mathrm{Hom}_{H,\chi}(V_1,\ V/\mathrm{Fil}_{k_0} V) = 0 \tag{2.2}$$

The short exact sequence $\mathrm{Fil}_{k_0-1}V \hookrightarrow \mathrm{Fil}_{k_0}V \twoheadrightarrow \mathrm{gr}_{k_0}V$ induces the long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0-1} V) &\to \mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0} V)\\ &\to \mathrm{Ext}^1_{H,\chi}\left(V_1,\ \mathrm{gr}_{k_0} V\right) \to \mathrm{Ext}^2_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0-1} V)\end{aligned}$$

which implies that

$$\dim_E \mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0} V) = 1 \tag{2.3}$$

by (2.1). The short exact sequence $\mathrm{Fil}_{k_0}V \hookrightarrow V \twoheadrightarrow V/\mathrm{Fil}_{k_0}V$ induces the long exact sequence

$$\begin{aligned}\mathrm{Hom}_{H,\chi}(V_1,\ V/\mathrm{Fil}_{k_0}V) &\to \mathrm{Ext}^1_{H,\chi}(V_1,\ \mathrm{Fil}_{k_0}V)\\ &\to \mathrm{Ext}^1_{H,\chi}(V_1,\ V) \to \mathrm{Ext}^1_{H,\chi}(V_1,\ V/\mathrm{Fil}_{k_0}V)\end{aligned}$$

which finishes the proof by combining (2.2) and (2.3). □

2.3. **Some representations of** $\mathrm{GL}_2(\mathbf{Q}_p)$ **and** $\mathrm{GL}_3(\mathbf{Q}_p)$**.** In this section, we are going to recall the construction of some locally analytic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\mathrm{GL}_3(\mathbf{Q}_p)$.

We denote the lower-triangular Borel subgroup (resp. the diagonal maximal split torus) of $\mathrm{GL}_{2/\mathbf{Q}_p}$ by B_2 (resp. by T_2) and the unipotent radical of B_2 by N_{GL_2}. We use the notation s for the non-trivial element in the Weyl group of GL_2. We fix a weight $\nu \in X(T_2)$ of GL_2 of the following form

$$\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$$

which corresponds to an algebraic character of $T_2(\mathbf{Q}_p)$

$$\delta_{T_2,\nu} \stackrel{\mathrm{def}}{=} \begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} \mapsto a^{\nu_1}b^{\nu_2}.$$

We denote the upper-triangular Borel subgroup of GL_2 by $\overline{B_2}$. If ν is dominant with respect to $\overline{B_2}$, namely if $\nu_1 \geq \nu_2$, we use the notation $\overline{L}_{\mathrm{GL}_2}(\nu)$ (resp. $L_{\mathrm{GL}_2}(-\nu)$) for the irreducible algebraic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ with highest weight ν (resp. $-\nu$) with respect to the positive roots determined by $\overline{B_2}$ (resp. B_2). In particular, $\overline{L}_{\mathrm{GL}_2}(\nu)$ and $L_{\mathrm{GL}_2}(-\nu)$ are the dual of each other. We use the shortened notation

$$I^{\mathrm{GL}_2}_{B_2}(\chi_{T_2}) \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Q}_p)}_{B_2(\mathbf{Q}_p)}\chi_{T_2}\right)^{\mathrm{an}}$$

for any locally analytic character χ_{T_2} of $T_2(\mathbf{Q}_p)$ and set

$$i^{\mathrm{GL}_2}_{B_2}(\chi_{T_2}) \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Q}_p)}_{B_2(\mathbf{Q}_p)}\chi^{\infty}_{T_2}\right)^{\infty} \otimes_E \overline{L}_{\mathrm{GL}_2}(\nu)$$

if $\chi_{T_2} = \delta_{T_2,\nu}\otimes_E\chi^{\infty}_{T_2}$ is locally algebraic where $\chi^{\infty}_{T_2}$ is a smooth character of $T_2(\mathbf{Q}_p)$. Then we define the locally analytic Steinberg representation (of weight ν) as well as the smooth Steinberg representation for $\mathrm{GL}_2(\mathbf{Q}_p)$ as follows

$$\mathrm{St}^{\mathrm{an}}_2(\nu) \stackrel{\mathrm{def}}{=} I^{\mathrm{GL}_2}_{B_2}(\delta_{T_2,\nu})/\overline{L}_{\mathrm{GL}_2}(\nu),\ \mathrm{St}^{\infty}_2 \stackrel{\mathrm{def}}{=} i^{\mathrm{GL}_2}_{B_2}(1_{T_2})/1_2$$

where 1_2 (resp. 1_{T_2}) denotes the trivial representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ (resp. of $T_2(\mathbf{Q}_p)$).

We denote the lower-triangular Borel subgroup (resp. the diagonal maximal split torus) of $\mathrm{GL}_{3/\mathbf{Q}_p}$ by B (resp. by T) and the unipotent radical of B by N. We write $\mathrm{Diag}(a,b,c) \in T(\mathbf{Q}_p)$ for the diagonal matrix with diagonal entries given by $a,b,c \in \mathbf{Q}_p^{\times}$. We fix a weight $\lambda \in X(T)$ of GL_3 of the following form

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3,$$

which corresponds to an algebraic character of $T(\mathbf{Q}_p)$ defined by

$$\delta_{T,\lambda}(\mathrm{Diag}(a,b,c)) \stackrel{\mathrm{def}}{=} a^{\lambda_1}b^{\lambda_2}c^{\lambda_3}.$$

We denote the center of GL_3 by Z and notice that $Z(\mathbf{Q}_p) \cong \mathbf{Q}_p^\times$. Hence the restriction of $\delta_{T,\lambda}$ to $Z(\mathbf{Q}_p)$ gives an algebraic character of $Z(\mathbf{Q}_p)$ defined by

$$\delta_{Z,\lambda}(\mathrm{Diag}(a,a,a)) \stackrel{\mathrm{def}}{=} a^{\lambda_1+\lambda_2+\lambda_3}.$$

We use the shortened notation

$$\mathrm{Ext}^i_{H,\lambda}(-,-) \stackrel{\mathrm{def}}{=} \mathrm{Ext}^i_{H,\delta_{Z,\lambda}}(-,-)$$

for each closed subgroup $H \subseteq \mathrm{GL}_3(\mathbf{Q}_p)$ that contains $Z(\mathbf{Q}_p)$. In particular, the notation

$$\mathrm{Ext}^i_{H,0}(-,-)$$

means (higher) extensions with trivial character of $Z(\mathbf{Q}_p)$. We denote the upper-triangular Borel subgroup of GL_3 by $\overline{B}$. If λ is dominant with respect to $\overline{B}$, namely if $\lambda_1 \geq \lambda_2 \geq \lambda_3$, we use the notation $\overline{L}(\lambda)$ (resp. $L(-\lambda)$) for the irreducible algebraic representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ with highest weight λ (resp. $-\lambda$) with respect to the positive roots determined by $\overline{B}$ (resp. B). In particular, $\overline{L}(\lambda)$ and $L(-\lambda)$ are dual of each other. We use the notation $P_1 \stackrel{\mathrm{def}}{=} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$ and $P_2 \stackrel{\mathrm{def}}{=} \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$ for the two standard maximal parabolic subgroups of GL_3 with unipotent radical N_1 and N_2 respectively, and the notation $\overline{P_i}$ for the opposite parabolic subgroup of P_i for each $i = 1, 2$. We set

$$L_i \stackrel{\mathrm{def}}{=} P_i \cap \overline{P_i}$$

and set s_i for the simple reflection in the Weyl group of L_i for each $i = 1, 2$. In particular, the Weyl group

$$W_{\mathrm{GL}_3} = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$$

of GL_3 can be lifted to a subgroup of GL_3. Each element $w \in W_{\mathrm{GL}_3}$ acts on $X(T)$ via the dot action

$$w \cdot \lambda \stackrel{\mathrm{def}}{=} w(\lambda + (2,1,0)) - (2,1,0).$$

We will usually use the shortened notation N_i for the set of $\mathbf{Q}_p$-points of N_i if this does not cause any ambiguity. We use the notation $M(-\lambda)$ for the Verma module in $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$ with highest weight $-\lambda$ (with respect to B) and simple quotient $L(-\lambda)$ for each $\lambda \in X(T)$ (not necessarily dominant). Similarly, we use the notation $M_i(-\lambda)$ for the parabolic Verma module in $\mathcal{O}^{\mathfrak{p}_i}_{\mathrm{alg}}$ with highest weight $-\lambda$ with respect to B (cf. Section 9.4 of [Hum08]). We define $\overline{L}_i(\lambda)$ as the irreducible algebraic representation of $L_i(\mathbf{Q}_p)$ with a highest weight λ dominant with respect to $\overline{B} \cap L_i$. For example, if $\lambda \in X(T)_+$, then we know that λ, $s_i \cdot \lambda$ and $s_is_{3-i} \cdot \lambda$ are dominant with respect to $\overline{B} \cap L_{3-i}$ for each $i = 1, 2$. We use the following notation for various parabolic inductions

$$I_B^{\mathrm{GL}_3}(\chi) \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)} \chi\right)^{\mathrm{an}}, \ I_{P_i}^{\mathrm{GL}_3}(\pi_i) \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}_{P_i(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)} \pi_i\right)^{\mathrm{an}}$$

if χ is an arbitrary locally analytic character of $T(\mathbf{Q}_p)$ and π_i is an arbitrary locally analytic representation of $L_i(\mathbf{Q}_p)$ for each $i = 1, 2$. Moreover, we use the notation

$$i_B^{\mathrm{GL}_3}(\chi) \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)} \chi^\infty\right)^\infty \otimes_E \overline{L}(\lambda), \ i_{P_i}^{\mathrm{GL}_3}(\pi_i) \stackrel{\mathrm{def}}{=} \left(\mathrm{Ind}_{P_i(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)} \pi_i^\infty\right)^\infty \otimes_E \overline{L}(\lambda)$$

for each $i = 1, 2$ if $\chi = \delta_{T,\lambda} \otimes_E \chi^\infty$ and $\pi_i = \overline{L}_i(\lambda) \otimes_E \pi_i^\infty$ are locally algebraic where χ^∞ (resp. π_i^∞) is a smooth representation of $T(\mathbf{Q}_p)$ (resp. of $L_i(\mathbf{Q}_p)$). We will also use similar notation for parabolic induction to Levi subgroups such as $I_{B\cap L_i}^{L_i}$ and $i_{B\cap L_i}^{L_i}$ for each $i=1,2$. Then we define the locally analytic (generalized) Steinberg representation (of weight λ) as well as the smooth (generalized) Steinberg representation for $\mathrm{GL}_3(\mathbf{Q}_p)$ by

$$\mathrm{St}_3^{\mathrm{an}}(\lambda) \stackrel{\mathrm{def}}{=} I_B^{\mathrm{GL}_3}(\delta_{T,\lambda})/\left(I_{P_1}^{\mathrm{GL}_3}(\overline{L}_1(\lambda))+I_{P_2}^{\mathrm{GL}_3}(\overline{L}_2(\lambda))\right),$$
$$\mathrm{St}_3^{\infty} \stackrel{\mathrm{def}}{=} i_B^{\mathrm{GL}_3}(1_T)/\left(i_{P_1}^{\mathrm{GL}_3}(1_{L_1})+i_{P_2}^{\mathrm{GL}_3}(1_{L_2})\right)$$

and

$$v_{P_i}^{\mathrm{an}}(\lambda) \stackrel{\mathrm{def}}{=} I_{P_i}^{\mathrm{GL}_3}(\overline{L}_i(\lambda))/\overline{L}(\lambda),\ \ v_{P_i}^{\infty} \stackrel{\mathrm{def}}{=} i_{P_i}^{\mathrm{GL}_3}(1_{L_i})/1_3$$

where 1_3 (resp. 1_{L_i}, resp. 1_T) is the trivial representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ (resp. of $L_i(\mathbf{Q}_p)$ for each $i=1,2$, resp. of $T(\mathbf{Q}_p)$). We write 1 for the trivial representation of $\mathbf{Q}_p^\times$ and define the following irreducible smooth representations of $L_1(\mathbf{Q}_p)$:

$$\begin{array}{rcl}\pi_{1,1}^{\infty} & \stackrel{\mathrm{def}}{=} & \mathrm{St}_2^{\infty}\otimes_E 1\\ \pi_{1,2}^{\infty} & \stackrel{\mathrm{def}}{=} & i_{B_2}^{\mathrm{GL}_2}\left(1\otimes_E|\cdot|^{-1}\right)\otimes_E|\cdot|\\ \pi_{1,3}^{\infty} & \stackrel{\mathrm{def}}{=} & \left(\mathrm{St}_2^{\infty}\otimes_E(|\cdot|^{-1}\circ\det{}_2)\right)\otimes_E|\cdot|^2\end{array}$$

and the following smooth representations of $L_2(\mathbf{Q}_p)$:

$$\begin{array}{rcl}\pi_{2,1}^{\infty} & \stackrel{\mathrm{def}}{=} & 1\otimes_E\mathrm{St}_2^{\infty}\\ \pi_{2,2}^{\infty} & \stackrel{\mathrm{def}}{=} & |\cdot|^{-1}\otimes_E i_{B_2}^{\mathrm{GL}_2}\left(|\cdot|\otimes_E 1\right)\\ \pi_{2,3}^{\infty} & \stackrel{\mathrm{def}}{=} & |\cdot|^{-2}\otimes_E\left(\mathrm{St}_2^{\infty}\otimes_E(|\cdot|\circ\det{}_2)\right)\end{array}$$

Consequently, we can define the following locally analytic representations for each $i=1,2$:

$$\begin{aligned}C_{s_i,1}^1 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_i\cdot\lambda),\ 1_{L_{3-i}}\right)\\ C_{s_i,1}^2 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_i\cdot\lambda),\ \pi_{3-i,1}^{\infty}\right)\\ C_{s_is_{3-i},1}^1 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_is_{3-i}\cdot\lambda),\ 1_{L_{3-i}}\right)\\ C_{s_is_{3-i},1}^2 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_is_{3-i}\cdot\lambda),\ \pi_{3-i,1}^{\infty}\right)\\ C_{s_i,s_i} &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_i\cdot\lambda),\ \pi_{3-i,2}^{\infty}\right)\\ C_{s_is_{3-i},s_i} &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_is_{3-i}\cdot\lambda),\ \pi_{3-i,2}^{\infty}\right)\\ C_{s_i,s_is_{3-i}}^1 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_i\cdot\lambda),\ \mathfrak{d}_{P_{3-i}}^{\infty}\right)\\ C_{s_i,s_is_{3-i}}^2 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_i\cdot\lambda),\ \pi_{3-i,3}^{\infty}\right)\\ C_{s_is_{3-i},s_is_{3-i}}^1 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_is_{3-i}\cdot\lambda),\ \mathfrak{d}_{P_{3-i}}^{\infty}\right)\\ C_{s_is_{3-i},s_is_{3-i}}^2 &\stackrel{\mathrm{def}}{=} \mathcal{F}_{P_{3-i}}^{\mathrm{GL}_3}\left(L(-s_is_{3-i}\cdot\lambda),\ \pi_{3-i,3}^{\infty}\right)\end{aligned}\tag{2.4}$$

where

$$\mathfrak{d}_{P_1}^{\infty} \stackrel{\mathrm{def}}{=} |\cdot|^{-1}\circ\det{}_2\otimes_E|\cdot|^2 \text{ and } \mathfrak{d}_{P_2}^{\infty} \stackrel{\mathrm{def}}{=} |\cdot|^{-2}\otimes_E|\cdot|\circ\det{}_2.$$

We also define

$$C_{s_1s_2s_1,w} \stackrel{\mathrm{def}}{=} \mathcal{F}_B^{\mathrm{GL}_3}\left(L(-s_1s_2s_1\cdot\lambda),\ \chi_w^{\infty}\right)\tag{2.5}$$

for each $w \in W_{\mathrm{GL}_3}$ where

$$\begin{aligned}
\chi_1^\infty &\stackrel{\mathrm{def}}{=} 1_T\\
\chi_{s_1}^\infty &\stackrel{\mathrm{def}}{=} |\cdot|^{-1}\otimes_E|\cdot|\otimes_E 1\\
\chi_{s_2}^\infty &\stackrel{\mathrm{def}}{=} 1\otimes_E|\cdot|^{-1}\otimes_E|\cdot|\\
\chi_{s_1s_2}^\infty &\stackrel{\mathrm{def}}{=} |\cdot|^{-2}\otimes_E|\cdot|\otimes_E|\cdot|\\
\chi_{s_2s_1}^\infty &\stackrel{\mathrm{def}}{=} |\cdot|^{-1}\otimes_E|\cdot|^{-1}\otimes_E|\cdot|^2\\
\chi_{s_1s_2s_1}^\infty &\stackrel{\mathrm{def}}{=} |\cdot|^{-2}\otimes_E 1\otimes_E|\cdot|^2
\end{aligned}$$

The simple objects in the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ can be described explicitly for each parabolic subgroup $P \subseteq \mathrm{GL}_3$, and the representations considered in (2.4) and (2.5) are all irreducible objects inside $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ according to the main theorem of [OS15]. We define Ω as the set that consists of $C_{s_1s_2s_1,w}$ for each $w \in W_{\mathrm{GL}_3}$, as well as the following elements:

$$\begin{array}{cccc}
\overline{L}(\lambda) & \overline{L}(\lambda)\otimes_E v_{P_1}^\infty & \overline{L}(\lambda)\otimes_E v_{P_2}^\infty & \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\\
C^1_{s_1,1} & C^2_{s_1,1} & C^1_{s_2,1} & C^2_{s_2,1}\\
C^1_{s_1s_2,1} & C^2_{s_1s_2,1} & C^1_{s_2s_1,1} & C^2_{s_2s_1,1}\\
C^1_{s_1,s_1s_2} & C^2_{s_1,s_1s_2} & C^1_{s_2,s_2s_1} & C^2_{s_2,s_2s_1}\\
C^1_{s_1s_2,s_1s_2} & C^2_{s_1s_2,s_1s_2} & C^1_{s_2s_1,s_2s_1} & C^2_{s_2s_1,s_2s_1}\\
C_{s_1,s_1} & C_{s_1s_2,s_1} & C_{s_2,s_2} & C_{s_2s_1,s_2}
\end{array} \tag{2.6}$$

Remark 2.3. The sets of Jordan–Hölder factors of various smooth parabolic inductions of χ_w^∞ and (parabolic) Verma modules of GL_3 are well known (cf. (48),(53) of [Bre17] and Section 9.5 of [Hum08] respectively). Then it follows quickly from the main theorem of [OS15] that

$$\Omega = \bigcup_{w\in W_{\mathrm{GL}_3}} \mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(I_B^{\mathrm{GL}_3}(\chi_w^\infty)\right).$$

Lemma 2.4. *The representation $v_{P_i}^{\mathrm{an}}(\lambda)$ fits into a non-split extension*

$$\overline{L}(\lambda)\otimes_E v_{P_i}^\infty \hookrightarrow v_{P_i}^{\mathrm{an}}(\lambda) \twoheadrightarrow C^1_{s_{3-i},1} \tag{2.7}$$

for each $i = 1, 2$. On the other hand, the representation $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ has the following form:

$$\begin{tikzcd}[column sep=small]
 & C^2_{s_1,1} \arrow[r, no head] \arrow[drr, no head] & C^1_{s_2s_1,1} \arrow[r, no head] & C^2_{s_2s_1,1} \arrow[dr, no head, dashed] & \\
\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \arrow[ur, no head] \arrow[dr, no head] & & & & C_{s_1s_2s_1,1}\,. \\
 & C^2_{s_2,1} \arrow[r, no head] \arrow[urr, no head] & C^1_{s_1s_2,1} \arrow[r, no head] & C^2_{s_1s_2,1} \arrow[ur, no head, dashed] &
\end{tikzcd} \tag{2.8}$$

Proof. The first claim follows directly from (3.62) of [BD20]. It follows from the main theorem of [OSc14] that

$$\begin{aligned}
&\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathrm{St}_3^{\mathrm{an}}(\lambda)\right)\\
&\quad = \{\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty,\ C^2_{s_1,1},\ C^2_{s_2,1},\ C^1_{s_2s_1,1},\ C^1_{s_1s_2,1},\ C^2_{s_2s_1,1},\ C^2_{s_1s_2,1},\ C_{s_1s_2s_1,1}\}
\end{aligned}$$

and each Jordan–Hölder factor occurs with multiplicity one. According to the fourth paragraph of the list before Corollaire 5.2.1 of [Bre17], we observe that

$$H_0\left(N_i,\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)\right) = \overline{L}_i(-s_{3-i}s_i\cdot\lambda)\otimes_E i_{B\cap L_i}^{L_i}(1_T)$$

which together with

$$\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)\right)=\{C^1_{s_{3-i}s_i,1},\ C^2_{s_{3-i}s_i,1}\} \tag{2.9}$$

implies that $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)$ fits into a non-split extension

$$C^1_{s_{3-i}s_i,1}\hookrightarrow\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)\twoheadrightarrow C^2_{s_{3-i}s_i,1} \tag{2.10}$$

for each $i=1,2$. Here (2.9) follows from the exactness of $\mathcal{F}_{P_i}^{\mathrm{GL}_3}$ and the irreducibility criterion in [OS15], as well as the fact that $i_{B\cap L_i}^{L_i}(1_T)$ has length two with Jordan–Hölder factors $\{1_{L_i},\ \pi_{i,1}^{\infty}\}$. According to Corollaire 5.3.2 as well as the list before Corollaire 5.2.1 of [Bre17], we observe that

$$H_2\left(N_{3-i},\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)\right)\ncong H_2(N_{3-i},\ C^2_{s_{3-i},1})\oplus H_2(N_{3-i},\ C^2_{s_{3-i}s_i,1})$$

which together with

$$\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)\right)=\{C^2_{s_{3-i},1},\ C^2_{s_{3-i}s_i,1}\} \tag{2.11}$$

implies that $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)$ fits into a non-split extension

$$C^2_{s_{3-i},1}\hookrightarrow\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)\twoheadrightarrow C^2_{s_{3-i}s_i,1} \tag{2.12}$$

for each $i=1,2$. Here (2.11) follows from the exactness of $\mathcal{F}_{P_i}^{\mathrm{GL}_3}$ and the irreducibility criterion in [OS15], as well as the fact that $M_i(-s_{3-i}\cdot\lambda)$ has length two with Jordan–Hölder factors $\{L_i(-s_{3-i}\cdot\lambda),\ L_i(-s_{3-i}s_i\cdot\lambda)\}$. We observe that both $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)$ and $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)$ are subquotients of $I_B^{\mathrm{GL}_3}(\delta_{T,\lambda})\cong\mathcal{F}_B^{\mathrm{GL}_3}\left(M(-\lambda),\ 1_T\right)$ (cf. [OS15]), and hence subquotients of $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ as well (using the fact that $\mathcal{F}_B^{\mathrm{GL}_3}\left(M(-\lambda),\ 1_T\right)$ is multiplicity free, which is a consequence of the main theorem of [OS15]). We finish the proof by combining (2.10) and (2.12) with the results before Remark 3.38 of [BD20]. □

Remark 2.5. One can show that all the possibly non-split extensions indicated in (2.8) are non-split. We decide not to go further here as Lemma 2.4 is precise enough for our application.

2.4. **p-adic logarithm and dilogarithm.** In this section, we recall the p-adic logarithm and dilogarithm function as well as their representation theoretical interpretations.

Let $\log_0\colon\mathbf{Q}_p^\times\to\mathbf{Q}_p$ be the branch of p-adic logarithm function which is given by the power series

$$\log_0(1+z)\overset{\mathrm{def}}{=}\sum_{k=0}^{\infty}\frac{z^k}{k}$$

on the open subgroup $1+p\mathbb{Z}_p\subseteq\mathbb{Z}_p^\times$ and satisfies the condition $\log_0(p)=\log_0(\zeta)=0$ for each root of unity ζ. Let $\mathrm{val}_p\colon\mathbf{Q}_p^\times\to\mathbb{Z}$ be the p-adic valuation function defined by $|\cdot|=p^{-\mathrm{val}_p(\cdot)}$ (hence $\mathrm{val}_p(p)=1$). We notice that

$$\{\log_0,\ \mathrm{val}_p\}$$

forms a basis of the two dimensional E-vector space

$$\mathrm{Hom}_{\mathrm{cont}}\left(\mathbf{Q}_p^\times,\ E\right).$$

We define $\log_{\mathscr{L}} \stackrel{\text{def}}{=} \log_0 - \mathscr{L}\mathrm{val}_p$ for each $\mathscr{L} \in E$ and consider the following two dimensional locally analytic representation of $\mathbf{Q}_p^\times$

$$V_{\mathscr{L}} \colon \mathbf{Q}_p^\times \to B_2(E),\ a \mapsto \begin{pmatrix} 1 & \log_{\mathscr{L}}(a) \\ 0 & 1 \end{pmatrix}.$$

We have

$$\mathrm{soc}_{\mathbf{Q}_p^\times}(V_{\mathscr{L}}) = \mathrm{cosoc}_{\mathbf{Q}_p^\times}(V_{\mathscr{L}}) = 1 \tag{2.13}$$

where 1 is the trivial character of $\mathbf{Q}_p^\times$. We notice that

$$\mathrm{Ext}^1_{\mathbf{Q}_p^\times}(1,1) \cong \mathrm{Hom}_{\mathrm{cont}}\left(\mathbf{Q}_p^\times,\ E\right),$$

by a standard fact in (continuous) group cohomology and therefore the set $\{V_{\mathscr{L}} \mid \mathscr{L} \in E\}$ exhausts (up to isomorphism) all different two dimensional locally analytic non-smooth E-representations of $\mathbf{Q}_p^\times$ satisfying (2.13). We abuse the notation $V_{\mathscr{L}}$ for the representation of $T_2(\mathbf{Q}_p) \cong \mathbf{Q}_p^\times \times \mathbf{Q}_p^\times$ given by composing with the map

$$T_2(\mathbf{Q}_p) \to \mathbf{Q}_p^\times,\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^{-1}b. \tag{2.14}$$

As a result, we can consider the parabolic induction

$$I_{B_2}^{\mathrm{GL}_2}\left(V_{\mathscr{L}} \otimes_E \delta_{T_2,\nu}\right)$$

which fits into an exact sequence (by exactness of $I_{B_2}^{\mathrm{GL}_2}$)

$$I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu}) \hookrightarrow I_{B_2}^{\mathrm{GL}_2}\left(V_{\mathscr{L}} \otimes_E \delta_{T_2,\nu}\right) \twoheadrightarrow I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu}). \tag{2.15}$$

Then we define $\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$ as the subrepresentation of $I_{B_2}^{\mathrm{GL}_2}\left(V_{\mathscr{L}} \otimes_E \delta_{T_2,\nu}\right) / \overline{L}_{\mathrm{GL}_2}(\nu)$ with cosocle $\overline{L}_{\mathrm{GL}_2}(\nu)$. It follows from (the proof of) Theorem 3.14 of [BD20] that $\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$ has the form

$$\mathrm{St}_2^{\mathrm{an}}(\nu) \longrightarrow \overline{L}_{\mathrm{GL}_2}(\nu) \tag{2.16}$$

and the set $\{\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L}) \mid \mathscr{L} \in E\}$ exhausts (up to isomorphism) all different locally analytic E-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ of the form (2.16) that do not contain

$$\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \longrightarrow \overline{L}_{\mathrm{GL}_2}(\nu)$$

as a subrepresentation. We have the embeddings

$$\iota_i \colon \mathrm{GL}_2 \hookrightarrow L_i$$

for each $i = 1, 2$ by identifying GL_2 with a Levi block of L_i, which induce the embeddings

$$\iota_{T,i} \colon T_2 \hookrightarrow T$$

by restricting ι_i to $T_2 \subsetneq \mathrm{GL}_2$. We use the notation $\iota_{T,i}(V_{\mathscr{L}})$ for the locally analytic representation of $T(\mathbf{Q}_p) \cong (\mathbf{Q}_p^\times)^3$ which is $V_{\mathscr{L}}$ after restricting to T_2 via $\iota_{T,i}$ and is trivial after restricting to the other copy of $\mathbf{Q}_p^\times$. By a direct analogue of $\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$, we can construct $\Sigma_{L_i}(\lambda,\mathscr{L})$ as the subrepresentation of $I_{B\cap L_i}^{L_i}\left(\iota_{T,i}(V_{\mathscr{L}}) \otimes_E \delta_{T,\lambda}\right) / \overline{L}_i(\lambda)$ with cosocle $\overline{L}_i(\lambda)$. In fact, if we have $\lambda|_{T_2,\iota_{T,i}} = \nu$, then we obviously know that $\Sigma_{L_i}(\lambda,\mathscr{L})|_{\mathrm{GL}_2,\iota_i} \cong \Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$ where the notation

$(\cdot)|_{*,\star}$ means the restriction of $\cdot$ to $*$ via the embedding $\star$. We observe that the parabolic induction $I_{P_i}^{\mathrm{GL}_3}(\Sigma_{L_i}(\lambda,\mathscr{L}))$ fits into the exact sequence

$$[\ v_{P_{3-i}}^{\mathrm{an}}(\lambda) \;\text{——}\; \mathrm{St}_3^{\mathrm{an}}(\lambda)\ ] \hookrightarrow I_{P_i}^{\mathrm{GL}_3}(\Sigma_{L_i}(\lambda,\mathscr{L})) \twoheadrightarrow [\ \overline{L}(\lambda) \;\text{——}\; v_{P_i}^{\mathrm{an}}(\lambda)\ ].$$

According to Proposition 5.6 of [Schr11], we know that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) = 0$$

and thus we can define $\Sigma_i(\lambda,\mathscr{L})$ as the unique quotient of $I_{P_i}^{\mathrm{GL}_3}(\Sigma_{L_i}(\lambda,\mathscr{L}))$ that fits into the exact sequence

$$\mathrm{St}_3^{\mathrm{an}}(\lambda) \hookrightarrow \Sigma_i(\lambda,\mathscr{L}) \twoheadrightarrow v_{P_i}^{\mathrm{an}}(\lambda).$$

We use the same notation $b_{i,\log_0}$ and b_{i,val_p} for the image of $\log_0$ and val_p respectively under the embedding

$$\mathrm{Ext}^1_{\mathbf{Q}_p^\times}(1,\ 1) \hookrightarrow \mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T,\ 1_T) \tag{2.17}$$

induced by the maps

$$T(\mathbf{Q}_p) \twoheadrightarrow T_2(\mathbf{Q}_p) \xrightarrow{(2.14)} \mathbf{Q}_p^\times$$

where the first map comes from the projection $L_i \twoheadrightarrow \mathrm{GL}_2$ by restriction to T. Hence the set

$$\{b_{1,\log_0},\ b_{1,\mathrm{val}_p},\ b_{2,\log_0},\ b_{2,\mathrm{val}_p}\} \tag{2.18}$$

forms a basis of $\mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T,\ 1_T)$. Recall the elements

$$c_{i,\log}, c_{i,\mathrm{val}} \in \mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T, 1_T)$$

constructed after (5.24) of [Schr11] and observe that

$$\begin{cases} c_{1,\log} = b_{1,\log_0} + 2b_{2,\log_0}, & c_{1,\mathrm{val}} = b_{1,\mathrm{val}_p} + 2b_{2,\mathrm{val}_p} \\ c_{2,\log} = 2b_{1,\log_0} + b_{2,\log_0}, & c_{2,\mathrm{val}} = 2b_{1,\mathrm{val}_p} + b_{2,\mathrm{val}_p} \end{cases}. \tag{2.19}$$

According to (5.70) and (5.71) of [Schr11], we notice that there exists canonical surjections

$$\mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T,\ 1_T) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_i}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) \tag{2.20}$$

with kernel spanned by $\{c_{i,\log},\ c_{i,\mathrm{val}}\}$. For each $i=1,2$, the previous constructions of $\Sigma_i(\lambda,\mathscr{L})$ can be explained by the composition

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{cont}}\left(\mathbf{Q}_p^\times,\ E\right) \\ &\quad \cong \mathrm{Ext}^1_{\mathbf{Q}_p^\times}(1,\ 1) \hookrightarrow \mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T,\ 1_T) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_i}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) \end{aligned} \tag{2.21}$$

with the second and third morphism given by (2.17) and (2.20) respectively. We deduce from (2.19) and the explicit description of (2.17) and (2.20) that the composition (2.21) is actually an isomorphism. We abuse the notation $b_{i,\log_0}$ and b_{i,val_p} for the image of $\log_0$ and val_p under the composition (2.21), and then notice that the image of $c_{3-i,\log}$ and $c_{3-i,\mathrm{val}}$ under (2.20) is given by $-3b_{i,\log_0}$ and $-3b_{i,\mathrm{val}_p}$ respectively.

We define $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ as the amalgamate sum of $\Sigma_1(\lambda, \mathscr{L}_1)$ and $\Sigma_2(\lambda, \mathscr{L}_2)$ over $\mathrm{St}_3^{\mathrm{an}}(\lambda)$, for each $\mathscr{L}_1, \mathscr{L}_2 \in E$. Consequently, $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ has the following form

$$\begin{tikzcd}[row sep=tiny] & v_{P_1}^{\mathrm{an}}(\lambda) \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & \\ & v_{P_2}^{\mathrm{an}}(\lambda) \end{tikzcd} . \tag{2.22}$$

In fact, if

$$\mathscr{L}_1 = -\mathscr{L}', \mathscr{L}_2 = -\mathscr{L} \in E, \tag{2.23}$$

we can identify our $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ with the $\Sigma(\lambda, \mathscr{L}, \mathscr{L}')$ in Definition 5.12 of [Schr11], defined using the element

$$(c_{2,\log} + \mathscr{L}' c_{2,\mathrm{val}},\ c_{1,\log} + \mathscr{L} c_{1,\mathrm{val}}) \in \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_1}^{\mathrm{an}}(\lambda) \oplus v_{P_2}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right).$$

Remark 2.6. In fact, one can identify $\mathscr{L}_1$ and $\mathscr{L}_2$ with Fontaine–Mazur $\mathscr{L}$-invariants of the corresponding Galois representation via local-global compatibility, according to Remark 3.1 of [Ding19]. This is the reason for the appearance of a sign in (2.23).

We have the following canonical morphism by (5.26) of [Schr11]

$$\kappa\colon \mathrm{Ext}^2_{T(\mathbf{Q}_p),0}(1_T, 1_T) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right). \tag{2.24}$$

Note that we also have

$$\mathrm{Ext}^2_{T(\mathbf{Q}_p),0}(1_T, 1_T) \cong \wedge^2 \left(\mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T, 1_T)\right) \tag{2.25}$$

by (5.24) of [Schr11]. The set

$$\{b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}, b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}, b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}, b_{1,\log_0} \wedge b_{2,\log_0}, b_{1,\mathrm{val}_p} \wedge b_{1,\log_0}, b_{2,\mathrm{val}_p} \wedge b_{2,\log_0}\} \tag{2.26}$$

forms a basis of $\wedge^2 \left(\mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T, 1_T)\right)$ (cf. (2.18)) and we abuse the same notation (2.26) for the corresponding basis of $\mathrm{Ext}^2_{T(\mathbf{Q}_p),0}(1_T,\ 1_T)$ (cf. (2.25)). It follows from (5.27) of [Schr11] and (2.19) that the set

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}), \kappa(b_{1,\log_0} \wedge b_{2,\log_0})\}$$

forms a basis of the image of (2.24).

Let $li_2\colon \mathbf{Q}_p \setminus \{0,1\} \to \mathbf{Q}_p$ be the p-adic dilogarithm function defined by Coleman in [Cole82] and we consider the function

$$D_{\mathscr{L}}(z) \overset{\mathrm{def}}{=} li_2(z) + \frac{1}{2}\log_{\mathscr{L}}(z)\log_{\mathscr{L}}(1-z)$$

as in (5.34) of [Schr11]. We also define

$$d(z) \overset{\mathrm{def}}{=} \log_0(1-z)\mathrm{val}_p(z) - \log_0(z)\mathrm{val}_p(1-z)$$

as in (5.36) of [Schr11] and it is clear that

$$D_{\mathscr{L}} - D_0 = \frac{\mathscr{L}}{2} d.$$

It follows from Theorem 7.2 of [Schr11] that $\{D_0, d\}$ can be interpreted as a basis of

$$\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p),0}(1,\ \mathrm{St}_2^{\mathrm{an}})$$

which naturally embeds into $\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}(1,\ \mathrm{St}_2^{\mathrm{an}})$ (cf. (5.37) and (5.38) of [Schr11]). Then the map $\iota_i\colon \mathrm{GL}_2 \hookrightarrow L_i$ induces the isomorphisms (cf. (5.42) of [Schr11])

(2.27)

$$\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}(1_2,\ \mathrm{St}_2^{\mathrm{an}}) \xleftarrow{\sim} \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),0}(1_{L_i},\ \mathrm{St}_2^{\mathrm{an}}) \xleftarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}\left(1_3,\ I_{P_i}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}})\right)$$

where $L_i(\mathbf{Q}_p)$ acts on $\mathrm{St}_2^{\mathrm{an}}$ via the projection $L_i(\mathbf{Q}_p) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Q}_p)$. We consider the following morphisms

(2.28)

$$\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}(1_2,\ \mathrm{St}_2^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}\left(1_3,\ I_{P_i}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}})\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}(1_3,\ \mathrm{St}_3^{\mathrm{an}})$$

induced by the inverse of the composition (2.27) as well as the surjection $I_{P_i}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}}) \twoheadrightarrow \mathrm{St}_3^{\mathrm{an}}$. Finally there is a canonical isomorphism

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}(1_3,\ \mathrm{St}_3^{\mathrm{an}}) \cong \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$$

by (5.20) of [Schr11].

Lemma 2.7. *We have*

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) = 5.$$

Proof. This follows directly from Proposition 5.6 of [Schr11]. □

Lemma 2.8. *There exists $\alpha \in E^\times$ such that*

$$\iota_1(d) = \iota_2(d) = -3\alpha\left(\kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p} + b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}\right).$$

Proof. This follows directly from Lemma 5.8 of [Schr11] and (2.19). □

Remark 2.9. It follows from the proof of Lemma 5.9 of [Schr11] that $\iota_1(D_0)-\iota_2(D_0)$ is a linear combination of

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}), \kappa(b_{1,\log_0} \wedge b_{2,\log_0})\},$$

but à priori we do not know the coefficients of this linear combination.

We recall from (5.55) of [Schr11] that

$$c_0 \overset{\mathrm{def}}{=} \alpha^{-1}\iota_1(D_0) - \frac{1}{2}\kappa(c_{1,\log} \wedge c_{2,\log}) \tag{2.29}$$

where α is defined in Lemma 5.8 of [Schr11].

Lemma 2.10. *The set*

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}), \kappa(b_{1,\log_0} \wedge b_{2,\log_0}), c_0\}$$

forms a basis of $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$.

Proof. This follows directly from (5.57) of [Schr11] and (2.19). □

Lemma 2.11. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = 1$$

and

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = 2.$$

Moreover, the image of

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), c_0\}$$

under

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

forms a basis of $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$.

Proof. This follows directly from Corollary 5.17 of [Schr11] and (2.19). □

3. A KEY RESULT FOR $\mathrm{GL}_2(\mathbf{Q}_p)$

The goal of this section is to prove Proposition 3.5, which is a key technical result that excludes the existence of a locally analytic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ with a specific form. Note that Proposition 3.5 will be crucially used in Section 5 and Section 6 (most notably in Proposition 5.4 and Proposition 6.2). We usually identify $\mathrm{GL}_2(\mathbf{Q}_p)$ with a Levi factor of a maximal parabolic of GL_3 when we apply the results from this section.

We use the following shortened notation

$$I(\nu) \stackrel{\mathrm{def}}{=} I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu}),\ \widetilde{I}(\nu) \stackrel{\mathrm{def}}{=} I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu} \otimes_E (|\cdot|^{-1} \otimes_E |\cdot|))$$

for each weight $\nu \in X(T_2)$.

Lemma 3.1. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\widetilde{I}(s\cdot\nu),\ \Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})\right) = 1.$$

Proof. This is essentially part of the proof of Theorem 3.14 of [BD20]. In fact, we know that

$$\begin{array}{rcl} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\widetilde{I}(s\cdot\nu),\ \ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ --- } I(s\cdot\nu)\right) & = & 0 \\ \mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\widetilde{I}(s\cdot\nu),\ \ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ --- } I(s\cdot\nu)\right) & = & 0 \end{array}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\widetilde{I}(s\cdot\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu)) = 1$$

which finish the proof by a simple dévissage induced by the short exact sequence

$$\left(\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ --- } I(s\cdot\nu)\ \right) \hookrightarrow \Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L}) \twoheadrightarrow \overline{L}_{\mathrm{GL}_2}(\nu).$$

□

For each split p-adic reductive group H, we have a natural embedding

$$U(\mathfrak{h}) \hookrightarrow D(H,E)_{\{1\}} \hookrightarrow D(H,E)$$

where $D(H,E)_{\{1\}}$ is the closed subalgebra of $D(H,E)$ consisting of distributions supported at the identity element (cf. [Koh07]). The embedding above induces another embedding

$$Z(U(\mathfrak{h})) \hookrightarrow Z(D(H,E)) \tag{3.1}$$

by the main result of [Koh07] where $Z(\cdot)$ is the notation for the center of an E-algebra. We say that $\Pi \in \mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_2(\mathbf{Q}_p),E}$ has an infinitesimal character if $Z(U(\mathfrak{h}))$ acts on Π' via a character.

Lemma 3.2. *If $V_1, V_2 \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ have both the same central character and the same infinitesimal character and satisfy*

$$\mathrm{Hom}_H(V_2,\ V_1) = 0,$$

then any non-split extension of the form $V_1 \longrightarrow V_2$ has both the same central character and the same infinitesimal character as the one for V_1 and V_2.

Proof. This is a direct analogue of Lemma 3.1 in [BD20] and follows essentially from the fact that both $D(Z(H), E)$ and $Z(U(\mathfrak{h}))$ are subalgebras of $Z(D(H, E))$ by [Koh07]. □

We fix a Borel subgroup $B_H \subseteq H$ as well as its opposite Borel subgroup $\overline{B_H}$. We consider the split maximal torus $T_H \stackrel{\mathrm{def}}{=} B_H \cap \overline{B_H}$ and use the notation N_H (resp. $\overline{N_H}$) for the unipotent radical of B_H (resp. of $\overline{B_H}$). We use the notation $J_{\overline{B_H}}(\cdot)$ for Emertion's Jacquet functor (cf. [Eme06]).

Lemma 3.3. *If $V \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ has an infinitesimal character, then $U(\mathfrak{t}_\mathfrak{h})^{W_H}$ (as a subalgebra of $U(\mathfrak{t}_\mathfrak{h})$) acts on $J_{\overline{B_H}}(V)$ via a character where W_H is the Weyl group of H.*

Proof. We know by our assumption that $Z(U(\mathfrak{h}))$ acts on V' (and hence on V as well) via a character. We note from (3.1) that $Z(U(\mathfrak{h}))$ commutes with $D(\overline{N_H}, E) \subseteq D(H, E)$ and thus the action of $Z(U(\mathfrak{h}))$ on V commutes with that of $\overline{N_H}$, which implies that $Z(U(\mathfrak{h}))$ acts on $V^{\overline{N_H}^\circ}$ via a character for each open compact subgroup $\overline{N_H}^\circ \subseteq \overline{N_H}$. We write

$$\theta\colon Z(U(\mathfrak{h})) \xrightarrow{\sim} U(\mathfrak{t}_\mathfrak{h})^{W_H}$$

for the Harish-Chandra isomorphism (cf. Section 1.7 of [Hum08]) and j_1 and j_2 for the embeddings

$$j_1\colon Z(U(\mathfrak{h})) \hookrightarrow U(\mathfrak{h}) \text{ and } j_2\colon U(\mathfrak{t}_\mathfrak{h}) \hookrightarrow U(\mathfrak{h}).$$

We choose an arbitrary Verma module $M_H(\lambda_H)$ with highest weight λ_H, namely we have

$$M_H(\lambda) \stackrel{\mathrm{def}}{=} U(\mathfrak{h}) \otimes_{U(\overline{\mathfrak{b}_H})} \lambda_H.$$

We use the notation $M_H(\lambda_H)_\mu$ for the subspace of $M_H(\lambda)$ with $\mathfrak{t}_\mathfrak{h}$-weight μ and note that

$$\dim_E M_H(\lambda_H)_{\lambda_H} = 1.$$

We easily observe that

$$Z(U(\mathfrak{h})) \cdot M_H(\lambda_H)_{\lambda_H} = M_H(\lambda_H)_{\lambda_H} \text{ and } U(\mathfrak{t}_\mathfrak{h}) \cdot M_H(\lambda_H)_{\lambda_H} = M_H(\lambda_H)_{\lambda_H}. \tag{3.2}$$

It is well-known that the direct sum decomposition

$$\mathfrak{h} = \mathfrak{n}_H \oplus \mathfrak{t}_\mathfrak{h} \oplus \overline{\mathfrak{n}_H} \tag{3.3}$$

induces a tensor decomposition of E-vector space

$$U(\mathfrak{h}) = U(\mathfrak{n}_H) \otimes_E U(\mathfrak{t}_\mathfrak{h}) \otimes_E U(\overline{\mathfrak{n}_H}). \tag{3.4}$$

Hence we can write each element in $U(\mathfrak{h})$ as a polynomial with variables indexed by a standard basis of $\mathfrak{h}$ that is compatible with (3.3). It follows from the definition of θ as the restriction to $Z(U(\mathfrak{h}))$ of the projection $U(\mathfrak{h}) \twoheadrightarrow U(\mathfrak{t}_\mathfrak{h})$ (coming from (3.4)) that

$$j_1(z) - j_2 \circ \theta(z) \in U(\mathfrak{h}) \cdot \overline{\mathfrak{n}_H} + \mathfrak{n}_H \cdot U(\mathfrak{h})$$

for each $z \in Z(U(\mathfrak{h}))$. If a monomial $f \neq 0$ in the decomposition (3.4) of $j_1(z) - j_2 \circ \theta(z)$ belongs to

$$\mathfrak{n}_H \cdot U(\mathfrak{n}_H) \cdot U(\mathfrak{t}_{\mathfrak{h}}),$$

then we have

$$0 \neq f \cdot M_H(\lambda_H)_{\lambda_H} \subseteq \mathfrak{n}_H \cdot M_H(\lambda_H)_{\lambda_H} \subseteq \bigoplus_{\mu \neq \lambda_H} M_H(\lambda_H)_{\mu},$$

which contradicts (3.2). Hence we conclude that

$$j_1(z) - j_2 \circ \theta(z) \in U(\mathfrak{h}) \cdot \overline{\mathfrak{n}_H}$$

and in particular

$$j_1(z) = j_2 \circ \theta(z)$$

on $V^{\overline{N_H}^\circ}$ for each $z \in Z(U(\mathfrak{h}))$. Hence we deduce that $U(\mathfrak{t}_{\mathfrak{h}})^{W_H}$ acts on $V^{\overline{N_H}^\circ}$ via a character. We note by the definition of $J_{\overline{B_H}}$ (cf. [Eme06]) that we have a T_H^+-equivariant embedding

$$J_{\overline{B_H}}(V) \hookrightarrow V^{\overline{N_H}^\circ} \tag{3.5}$$

where T_H^+ is a certain submonoid of T_H containing an open compact subgroup. As a result, (3.5) is also $U(\mathfrak{t}_{\mathfrak{h}})$-equivariant and thus $U(\mathfrak{t}_{\mathfrak{h}})^{W_H}$ acts on $J_{\overline{B_H}}(V)$ via a character which finishes the proof. □

We take $H = \mathrm{GL}_2(\mathbf{Q}_p)$, $B_H = B_2$ and $\overline{B_H} = \overline{B_2}$ in the rest of this section. The idea of the following lemma which is closely related to Lemma 3.20 of [BD20], owes very much to Y.Ding.

Lemma 3.4. *A locally analytic representation of either the form*

$$\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;—\; I(s\cdot\nu) \;—\; \overline{L}_{\mathrm{GL}_2}(\nu) \;—\; \overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \tag{3.6}$$

or the form

$$\overline{L}_{\mathrm{GL}_2}(\nu) \;—\; \widetilde{I}(s\cdot\nu) \;—\; \overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;—\; \overline{L}_{\mathrm{GL}_2}(\nu) \tag{3.7}$$

does not have an infinitesimal character.

Proof. Assume that a representation V of the form (3.6) has an infinitesimal character. Note that V can be represented by an element in the space

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty, \Sigma_{\mathrm{GL}_2}(\nu, \mathscr{L}))$$

for certain $\mathscr{L} \in E$. We consider the upper-triangular Borel subgroup $\overline{B_2}$ and the diagonal split torus T_2. Then by the proof of Lemma 3.20 of [BD20] we know that the Jacquet functor $J_{\overline{B_2}}$ (cf. [Eme06] for the definition) induces a injection

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty,\ \Sigma_{\mathrm{GL}_2}(\nu, \mathscr{L})\right) \\ &\quad \hookrightarrow \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}\left(\delta_{T_2,\nu} \otimes_E (|\cdot| \otimes_E |\cdot|^{-1}),\ \delta_{T_2,\nu} \otimes_E (|\cdot| \otimes_E |\cdot|^{-1})\right). \end{aligned} \tag{3.8}$$

We deduce by twisting $\delta_{T_2,-\nu} \otimes_E (|\cdot|^{-1} \otimes_E |\cdot|)$ that we have an isomorphism

$$\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}\left(\delta_{T_2,\nu} \otimes_E (|\cdot| \otimes_E |\cdot|^{-1}),\ \delta_{T_2,\nu} \otimes_E (|\cdot| \otimes_E |\cdot|^{-1})\right) \cong \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2},\ 1_{T_2}). \tag{3.9}$$

It follows from Lemma 3.20 of [BD20] (up to changes on notation) that the image of the composition of (3.9) and (3.8) is a certain three dimensional subspace

$\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})_{\mathscr{L}}$ of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})$ depending on $\mathscr{L}$. More precisely, if we use the notation ϵ_1, ϵ_2 for the two characters

$$\epsilon_1\colon T_2(\mathbf{Q}_p) \to \mathbf{Q}_p^\times, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a \text{ and } \epsilon_2\colon T_2(\mathbf{Q}_p) \to \mathbf{Q}_p^\times, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto b,$$

then the set

$$\{\log_0 \circ \epsilon_1, \mathrm{val}_p \circ \epsilon_1, \log_0 \circ \epsilon_2, \mathrm{val}_p \circ \epsilon_2\}$$

forms a basis of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})$, and the subspace $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})_{\mathscr{L}}$ has

$$\{\log_0 \circ \epsilon_1 + \log_0 \circ \epsilon_2, \mathrm{val}_p \circ \epsilon_1 + \mathrm{val}_p \circ \epsilon_2, \log_0 \circ \epsilon_1 - \log_0 \circ \epsilon_2 + \mathscr{L}(\mathrm{val}_p \circ \epsilon_1 - \mathrm{val}_p \circ \epsilon_2)\}$$

as a basis. It follows from Lemma 3.3 that $U(\mathfrak{t}_2)^{W_{\mathrm{GL}_2}}$ acts on $J_{\overline{B_2}}(V)$ via a character where W_{GL_2} is the Weyl group of GL_2. Note that the subspace of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})$ corresponding to $J_{\overline{B_2}}(V)$ (by twisting $\delta_{T_2,-\nu} \otimes_E (|\cdot|^{-1} \otimes_E |\cdot|)$) is killed by $U(\mathfrak{t}_2)^{W_{\mathrm{GL}_2}}$. We observe that the subspace M of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})$ killed by $U(\mathfrak{t}_2)^{W_{\mathrm{GL}_2}}$ is two dimensional with basis

$$\{\mathrm{val}_p \circ \epsilon_1, \mathrm{val}_p \circ \epsilon_2\}$$

and we have

$$M \cap \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2}, 1_{T_2})_{\mathscr{L}} = E\left(\mathrm{val}_p \circ \epsilon_1 + \mathrm{val}_p \circ \epsilon_2\right).$$

However, the representation associated with the line $E(\mathrm{val}_p \circ \epsilon_1 + \mathrm{val}_p \circ \epsilon_2)$ has a subrepresentation of the form

$$\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;\text{---}\; \overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty$$

which contradicts the fact that V has the form (3.6).

The proof of the second statement is a direct analogue as we observe that $J_{\overline{B_2}}$ also induces the following embedding

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu) \;\text{---}\; \widetilde{I}(s\cdot\nu) \;\text{---}\; \overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;\text{---}\; \overline{L}_{\mathrm{GL}_2}(\nu)\Big)$$
$$\hookrightarrow \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}\left(\delta_{T_2,\nu},\ \delta_{T_2,\nu}\right).$$

□

We define $\Sigma_2^+(\nu, \mathscr{L})$ as the unique (up to isomorphism) non-split extension of $\Sigma_{\mathrm{GL}_2}(\nu, \mathscr{L})$ by $\widetilde{I}(s\cdot\nu)$ given by Lemma 3.1.

Proposition 3.5. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;\text{---}\; \overline{L}_{\mathrm{GL}_2}(\nu),\ \Sigma_2^+(\nu, \mathscr{L})\Big) = 0.$$

Proof. Assume on the contrary that V is a representation given by a certain non-zero element inside

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;\text{---}\; \overline{L}_{\mathrm{GL}_2}(\nu),\ \Sigma_2^+(\nu, \mathscr{L})\Big).$$

We deduce that V has both a central character and an infinitesimal character from Lemma 3.2 and the fact

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \;\text{---}\; \overline{L}_{\mathrm{GL}_2}(\nu),\ \Sigma_2^+(\nu, \mathscr{L})\Big) = 0.$$

As we have

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty,\ I(s\cdot\nu)) = \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\overline{L}_{\mathrm{GL}_2}(\nu),\ \widetilde{I}(s\cdot\nu)) = 0,$$

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\overline{L}_{\mathrm{GL}_2}(\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty\right) = 1$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\overline{L}_{\mathrm{GL}_2}(\nu),\ I(s\cdot\nu)\right) = 1$$

by a combination of Lemma 3.13 of [BD20] with Lemma 2.1, we deduce that V has a subrepresentation of one of the three following forms

(i) $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$;

(ii) $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ —— $I(s\cdot\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$;

(iii) $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ —— $I(s\cdot\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)$ —— $\widetilde{I}(s\cdot\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ — $\overline{L}_{\mathrm{GL}_2}(\nu)$.

In the first case, we know from Proposition 4.7 of [Schr11] and the main result of [Or05] that

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p),\nu}\left(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty,\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty\right) = 0$$

and therefore this case is impossible due to the existence of central character for V (and hence for its subrepresentations). In the second case, we deduce from Lemma 3.4 a contradiction as V has an infinitesimal character. In the third case, we thus know that V has a quotient representation of the form

$$\overline{L}_{\mathrm{GL}_2}(\nu) \text{ —— } \widetilde{I}(s\cdot\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)$$

which can not have an infinitesimal character due to Lemma 3.4, a contradiction again. Hence we finish the proof. □

Remark 3.6. Note that the argument in Proposition 3.5 actually implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\ ,\ I(s\cdot\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu) \text{ —— } \widetilde{I}(s\cdot\nu)\ \Big) = 0$$

and we can show by the same method that

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\ \overline{L}_{\mathrm{GL}_2}(\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty\ ,\ \widetilde{I}(s\cdot\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ —— } I(s\cdot\nu)\ \Big) = 0.$$

4. Computations of Ext I

In this section, we are going to compute a list of Ext-groups based on known results on group cohomology in Théorème 4.10 of [Schr11] and Section 5.2, 5.3 of [Bre17]. The technical results proved in this section will be frequently used in more complicated computation in Section 5 and Section 6. In each proposition or lemma below, we present a list of Ext-groups whose computations are parallel to each other.

Proposition 4.1. *The following E-vector spaces are one dimensional*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\right)$$
$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \overline{L}(\lambda)\right)$$
$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\right)$$
$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty,\ \overline{L}(\lambda)\right)$$
$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty\right)$$
$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty\right)$$

for each $i=1,2$. Moreover, for all the other choices of $V_1,V_2\in\{\overline{L}(\lambda),\overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\}$, we have

$$\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_1,\ V_2)=0$$

for each $k=1,2$.

Proof. This follows from a special case of Proposition 4.7 of [Schr11] and the main result of [Or05]. □

Lemma 4.2. *We have*

$$\begin{aligned}
\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \;\text{---}\; \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) &= 0\\
\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \;\text{---}\; \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\ ,\ \overline{L}(\lambda)\right) &= 0\\
\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda) \;\text{---}\; \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\ ,\ \overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty\right) &= 0
\end{aligned}$$

for each $i=1,2$ and $k=1,2$.

Proof. It is sufficient to prove that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \;\text{---}\; \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)=0 \tag{4.1}$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \;\text{---}\; \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)=0 \tag{4.2}$$

as the other cases are similar. We observe that (4.1) is equivalent to the non-existence of a representation of the form

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \;\text{---}\; \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \;\text{---}\; \overline{L}(\lambda)$$

which is again equivalent to the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \;\text{---}\; \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\ \right)=0, \tag{4.3}$$

using the fact (cf. Proposition 4.1)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)=0.$$

The short exact sequence

$$\left(\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \;\text{---}\; \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\ \right)\hookrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-\lambda),\ \pi_{1,3}^\infty\right)\twoheadrightarrow C^2_{s_{3-i},s_{3-i}s_i}$$

induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \text{ --- } \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\right)$$
$$\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-\lambda),\ \pi_{i,3}^\infty\right)\right).$$

Therefore (4.3) follows from Lemma 2.1 and the fact (using Théorème 4.10 of [Schr11] and a comparison of $Z(L_i(\mathbf{Q}_p))$-action)

$$\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(H_0(N_i,\overline{L}(\lambda)),\ \overline{L}_i(\lambda)\otimes_E \pi_{i,3}^\infty\right)$$
$$= \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(H_1(N_i,\overline{L}(\lambda)),\ \overline{L}_i(\lambda)\otimes_E \pi_{i,3}^\infty\right) = 0.$$

On the other hand, the short exact sequence

$$\overline{L}(\lambda)\otimes_E v_{P_i}^\infty \hookrightarrow \left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ --- } \overline{L}(\lambda)\ \right) \twoheadrightarrow \overline{L}(\lambda)$$

induces a long exact sequence

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ --- } \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ --- } \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$

and thus we can deduce (4.2) from Proposition 4.1 and (4.1). □

According to Proposition 4.1, we may define W_0 as the unique (up to isomorphism) locally algebraic representation of length three satisfying

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(W_0) = \overline{L}(\lambda)\otimes_E \left(v_{P_1}^\infty \oplus v_{P_2}^\infty\right) \text{ and } \mathrm{cosoc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(W_0) = \overline{L}(\lambda).$$

We also define the unique (up to isomorphism) locally algebraic representation of the form

$$W_i \overset{\mathrm{def}}{=} \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ --- } \overline{L}(\lambda) \tag{4.4}$$

for each $i = 1, 2$

Lemma 4.3. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 1$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 0.$$

Proof. The short exact sequence

$$\overline{L}(\lambda)\otimes_E v_{P_1}^\infty \hookrightarrow W_0 \twoheadrightarrow W_2$$

induces a long exact sequence

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$
$$\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)$$

which finishes the proof by Proposition 4.1, (4.1) and (4.2). □

Recall that we have introduced a set Ω consisting of irreducible locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ in (2.6). We define the following subsets of Ω:

$$\begin{array}{lll}
\Omega_1\left(\overline{L}(\lambda)\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C_{s_1,1}^1,\ C_{s_2,1}^1\} \\
\Omega_1\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C_{s_1,1}^2,\ C_{s_2,s_2},\ C_{s_1,s_1s_2}^1\} \\
\Omega_1\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C_{s_2,1}^2,\ C_{s_1,s_1},\ C_{s_2,s_2s_1}^1\} \\
\Omega_1\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C_{s_1,s_1s_2}^2,\ C_{s_2,s_2s_1}^2\} \\
\Omega_2\left(\overline{L}(\lambda)\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C_{s_1,1}^2,\ C_{s_2,1}^2,\ C_{s_1s_2,1}^1,\ C_{s_2s_1,1}^1\} \\
\Omega_2\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C_{s_1,1}^1,\ C_{s_1,s_1s_2}^2,\ C_{s_1s_2,1}^2,\ C_{s_2s_1,s_2}\} \\
\Omega_2\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ C_{s_2,1}^1,\ C_{s_2,s_2s_1}^2,\ C_{s_2s_1,1}^2,\ C_{s_1s_2,s_1}\} \\
\Omega_2\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) & \stackrel{\mathrm{def}}{=} & \{\overline{L}(\lambda),\ C_{s_1,s_1s_2}^1,\ C_{s_2,s_2s_1}^1,\ C_{s_1s_2,s_1s_2}^2,\ C_{s_2s_1,s_2s_1}^2\}
\end{array}$$

Lemma 4.4. *For each*

$$V_0\in\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\},$$

we have

$$\begin{cases}\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=1 & \textit{if} \quad V\in\Omega_1(V_0);\\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=0 & \textit{if} \quad V\in\Omega\setminus\Omega_1(V_0).\end{cases}$$

Proof. We only prove the statements for $V_0=\overline{L}(\lambda)$ as other cases are similar. If

$$V\in\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\}$$

then the conclusion follows from Proposition 4.1. If

$$V=\mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^{\infty})$$

for a smooth irreducible representation π_i^{∞} and $i=1$ or 2, then it follows from Lemma 2.1 that

$$\begin{aligned}(4.5)\quad & \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(H_0(N_i,\ \overline{L}(\lambda)),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\\ &\to \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(H_1(N_i,\ \overline{L}(\lambda)),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right)\\ &\to \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),\lambda}\left(H_0(N_i,\ \overline{L}(\lambda)),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right).\end{aligned}$$

We combine (4.5) with Théorème 4.10 of [Schr11] and deduce that

$$\begin{aligned}(4.6)\quad & \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\\ &\to \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right).\end{aligned}$$

We notice that $Z(L_i(\mathbf{Q}_p))$ acts via different characters on $\overline{L}_i(\lambda)$, $\overline{L}_i(s_{3-i}\cdot\lambda)$ and $\overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}$, and thus we have the equalities

$$\begin{array}{rcl}\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right) & = & 0\\ \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^{\infty}\right) & = & 0\end{array}$$

which imply that

$$(4.7)\qquad \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^{\infty})\right)=0$$

for each π_i^∞ and $i = 1, 2$. If

$$V = \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)$$

for a smooth irreducible representation π_i^∞ and $i = 1$ or 2, then the short exact sequence

$$\mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty) \hookrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty) \twoheadrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^\infty)$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right) &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^\infty)\right)\end{aligned}$$

which implies an isomorphism

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right) \tag{4.8}$$

by (4.7). It follows from (4.8), Théorème 4.10 of [Schr11] and Lemma 2.1 that

$$\begin{aligned}\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E\pi_i^\infty\right) &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\\ &\to \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E\pi_i^\infty\right)\\ &\to \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E\pi_i^\infty\right).\end{aligned} \tag{4.9}$$

As $Z(L_i(\mathbf{Q}_p))$ acts via different characters on $\overline{L}_i(\lambda)$ and $\overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E\pi_i^\infty$, we have the equalities

$$\begin{aligned}\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^\infty\right) &= 0\\ \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}s_i\cdot\lambda)\otimes_E\pi_i^\infty\right) &= 0\end{aligned}$$

which imply that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right) \xrightarrow{\sim} \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E\pi_i^\infty\right). \tag{4.10}$$

Note that

$$\mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E\pi_i^\infty\right) = 0$$

for each smooth irreducible $\pi_i^\infty \neq 1_{L_i}$. Hence we deduce that

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ 1_{L_i})\right) = 1$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right) = 0$$

for each smooth irreducible $\pi_i^\infty \neq 1_{L_i}$. Finally, similar methods together with Théorème 4.10 of [Schr11] also show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{B}^{\mathrm{GL}_3}(L(-s_1s_2s_1\cdot\lambda),\ \chi_w^\infty)\right) = 0$$

for each $w \in W$. □

Lemma 4.5. *For each*

$$V_0 \in \{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\},$$

we have

$$\begin{cases} \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V) = 1 & \text{if} \quad V \in \Omega_2(V_0); \\ \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V) = 0 & \text{if} \quad V \in \Omega \setminus \Omega_2(V_0). \end{cases}$$

Proof. We only prove the statements for $V_0 = \overline{L}(\lambda)$ as other cases are similar. If

$$V \in \{\overline{L}(\lambda),\ \overline{L}(\lambda) \otimes_E v^\infty_{P_1},\ \overline{L}(\lambda) \otimes_E v^\infty_{P_2},\ \overline{L}(\lambda) \otimes_E \mathrm{St}^\infty_3\}$$

then the conclusion follows from Proposition 4.1. We notice that $Z(L_i(\mathbf{Q}_p))$ acts via different characters on $\overline{L}_i(\lambda)$, $\overline{L}_i(s_{3-i} \cdot \lambda)$ and $\overline{L}_i(s_{3-i}s_i \cdot \lambda) \otimes_E \pi_i^\infty$, and thus we have

$$\begin{aligned} \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}s_i \cdot \lambda) \otimes_E \pi_i^\infty\right) &= 0 \\ \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i} \cdot \lambda),\ \overline{L}_i(s_{3-i}s_i \cdot \lambda) \otimes_E \pi_i^\infty\right) &= 0 \\ \mathrm{Ext}^3_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i}s_i \cdot \lambda) \otimes_E \pi_i^\infty\right) &= 0 \end{aligned} \tag{4.11}$$

We also notice that

$$\mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}s_i \cdot \lambda),\ \overline{L}_i(s_{3-i}s_i \cdot \lambda) \otimes_E \pi_i^\infty\right) = 0 \tag{4.12}$$

for each smooth irreducible $\pi_i^\infty \neq 1_{L_i}$ and

$$\dim_E \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}s_i \cdot \lambda),\ \overline{L}_i(s_{3-i}s_i \cdot \lambda)\right) = 1. \tag{4.13}$$

We combine (4.11), (4.12) and (4.13) with Lemma 2.1 and Théorème 4.10 of [Schr11] and deduce that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}^{\mathrm{GL}_3}_{P_i}(L(-s_{3-i}s_i \cdot \lambda),\ \pi_i^\infty)\right) = 0 \tag{4.14}$$

for each smooth irreducible $\pi_i^\infty \neq 1_{L_i}$ and

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}^{\mathrm{GL}_3}_{P_i}(L(-s_{3-i}s_i \cdot \lambda),\ 1_{L_i})\right) = 1 \tag{4.15}$$

which finishes the proof if

$$V = \mathcal{F}^{\mathrm{GL}_3}_{P_i}(L(-s_{3-i}s_i \cdot \lambda),\ \pi_i^\infty).$$

Similarly, we have

$$\begin{aligned} \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i} \cdot \lambda) \otimes_E \pi_i^\infty\right) &= 0 \\ \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}s_i \cdot \lambda),\ \overline{L}_i(s_{3-i} \cdot \lambda) \otimes_E \pi_i^\infty\right) &= 0 \\ \mathrm{Ext}^3_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(\lambda),\ \overline{L}_i(s_{3-i} \cdot \lambda) \otimes_E \pi_i^\infty\right) &= 0 \end{aligned} \tag{4.16}$$

We claim that

$$\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i} \cdot \lambda),\ \overline{L}_i(s_{3-i} \cdot \lambda) \otimes_E \pi_i^\infty\right) \cong \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),0}\left(1_{L_i},\ \pi_i^\infty\right)^{\mathrm{sm}} \tag{4.17}$$

for each $\pi_i^\infty \neq 1_{L_i}$, where the RHS means Ext^1 inside the abelian category $\mathrm{Rep}^\infty_{L_i(\mathbf{Q}_p),E}$. The reason behind (4.17) is that any non-split extension in LHS of (4.17) necessarily has infinitesimal character (using Lemma 3.2), hence must split after restricting to $\mathfrak{l}_i$. In other words, any non-split extension in LHS of (4.17) must have the form $\overline{L}_i(s_{3-i} \cdot \lambda) \otimes_E W$ where W is a smooth non-split extension coming from RHS of (4.17). Hence it is clear that

$$\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i} \cdot \lambda),\ \overline{L}_i(s_{3-i} \cdot \lambda) \otimes_E \pi_i^\infty\right) = 0 \tag{4.18}$$

for each smooth irreducible $\pi_i^\infty \neq 1_{L_i},\ \pi_{i,1}^\infty$ and

$$\dim_E \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\otimes_E \pi_{i,1}^\infty\right) = 1. \tag{4.19}$$

By adapting arguments in Section 4.2 (cf. (4.23) and Proposition 4.5) of [Schr11], we claim that

$$\begin{gathered}\dim_E \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\right) = 1,\\ \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),\lambda}\left(\overline{L}_i(s_{3-i}\cdot\lambda),\ \overline{L}_i(s_{3-i}\cdot\lambda)\right) = 0.\end{gathered} \tag{4.20}$$

We combine (4.16) and (4.18) with Lemma 2.1 and Théorème 4.10 of [Schr11] and deduce that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right) = 0 \tag{4.21}$$

for each smooth irreducible $\pi_i^\infty \neq 1_{L_i},\ \pi_{i,1}^\infty$. Similarly, we use (4.19) and (4.20) to conclude that

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right) = 1 \tag{4.22}$$

for $\pi_i^\infty = 1_{L_i},\ \pi_{i,1}^\infty$. The short exact sequence

$$\mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty) \hookrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty) \twoheadrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^\infty)$$

induces a long exact sequence

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^\infty)\right)\\ &\qquad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right)\\ &\qquad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(M_i(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right)\\ &\qquad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}s_i\cdot\lambda),\ \pi_i^\infty)\right)\\ &\qquad\qquad\to \mathrm{Ext}^3_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty)\right).\end{aligned}$$

The first term always vanishes thanks to Lemma 4.4. According to (4.14), the fourth terms vanishes whenever $\pi_i^\infty \neq 1_{L_i}$. If $\pi_i^\infty \neq 1_{L_i},\ \pi_{i,1}^\infty$, then the third term vanishes (cf. (4.21)), and so does the second term. If $\pi_i^\infty = \pi_{i,1}^\infty$, then the third terms has dimension one, and so does the second term. If $\pi_i^\infty = 1_{L_i}$, we note that the fifth term vanishes and both the third and fourth term have dimension one (cf. (4.20) and (4.22)), and thus the second term vanishes. Consequently, we finish the proof if

$$V = \mathcal{F}_{P_i}^{\mathrm{GL}_3}(L(-s_{3-i}\cdot\lambda),\ \pi_i^\infty).$$

Finally, similar methods together with Théorème 4.10 of [Schr11] also show that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_B^{\mathrm{GL}_3}(L(-s_1s_2s_1\cdot\lambda),\ \chi_w^\infty)\right) = 0$$

for each $w \in W$. $\square$

We define

$$\Omega^- \overset{\mathrm{def}}{=} \Omega \setminus \{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\}.$$

Then we define the following subsets of Ω^- for each $i=1,2$:

$$\begin{array}{lcl}
\Omega_1\left(C^1_{s_i,1}\right) & \stackrel{\mathrm{def}}{=} & \{C^1_{s_is_{3-i},1},\ C^2_{s_{3-i}s_i,1},\ C^2_{s_i,1},\ C^1_{s_i,1}\} \\
\Omega_1\left(C^2_{s_i,1}\right) & \stackrel{\mathrm{def}}{=} & \{C^2_{s_is_{3-i},1},\ C_{s_{3-i}s_i,s_{3-i}},\ C^1_{s_i,1},\ C^2_{s_i,1}\} \\
\Omega_1\left(C^1_{s_i,s_is_{3-i}}\right) & \stackrel{\mathrm{def}}{=} & \{C^1_{s_is_{3-i},s_is_{3-i}},\ C_{s_{3-i}s_i,s_{3-i}},\ C^2_{s_i,s_is_{3-i}},\ C^1_{s_i,s_is_{3-i}}\} \\
\Omega_1\left(C^2_{s_i,s_is_{3-i}}\right) & \stackrel{\mathrm{def}}{=} & \{C^2_{s_is_{3-i},s_is_{3-i}},\ C^1_{s_{3-i}s_i,s_{3-i}s_i},\ C^1_{s_i,s_is_{3-i}},\ C^2_{s_i,s_is_{3-i}}\} \\
\Omega_1\left(C_{s_i,s_i}\right) & \stackrel{\mathrm{def}}{=} & \{C_{s_is_{3-i},s_i},\ C^1_{s_{3-i}s_i,1},\ C^2_{s_{3-i}s_i,s_{3-i}s_i},\ C_{s_i,s_i}\}
\end{array}$$

Lemma 4.6. *For each*

$$V_0\in\{C^1_{s_i,1},\ C^2_{s_i,1},\ C^1_{s_i,s_is_{3-i}},\ C^2_{s_i,s_is_{3-i}},\ C_{s_i,s_i}\mid i=1,2\},$$

we have

$$\begin{cases} \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=1 & \textit{if}\quad V\in\Omega_1(V_0); \\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=0 & \textit{if}\quad V\in\Omega^-\setminus\Omega_1(V_0). \end{cases}$$

Proof. The proof is very similar to that of Lemma 4.4, and the main difference is that we need Corollaire 5.3.2 of [Bre17] instead of the list before Corollaire 5.2.1 of [Bre17]. □

Lemma 4.7. *We have*

$$\begin{array}{ll}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{---} \overline{L}(\lambda)\ ,\ C^2_{s_i,1}\right) & =0 \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{---} \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\ ,\ C^1_{s_i,s_is_{3-i}}\right) & =0 \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \text{---} \overline{L}(\lambda)\otimes_E v^\infty_{P_i}\ ,\ C^1_{s_i,1}\right) & =0 \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3 \text{---} \overline{L}(\lambda)\otimes_E v^\infty_{P_i}\ ,\ C^2_{s_i,s_is_{3-i}}\right) & =0
\end{array}$$

for each $i=1,2$.

Proof. We recall the shortened notation W_i from (4.4) and note from (53) of [Bre17] that $W_i\cong i^{\mathrm{GL}_3}_{P_{3-i}}(\mathfrak{d}^\infty_{P_{3-i}})$ for each $i=1,2$ (cf. Section 2.3 for the notation $i^{\mathrm{GL}_3}_{P_{3-i}}(\cdot)$ and $\mathfrak{d}^\infty_{P_{3-i}}$). We only prove the first vanishing (among four)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ C^2_{s_i,1}\right)=0 \tag{4.23}$$

as the other cases are similar. The embedding

$$C^2_{s_i,1}\hookrightarrow \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^\infty_{3-i,1})$$

induces an embedding (using a vanishing of Hom)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ C^2_{s_i,1}\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^\infty_{3-i,1})\right). \tag{4.24}$$

We observe from (48) as well as the first paragraph of the list before Corollaire 5.2.1 of [Bre17] that

$$\begin{array}{lcl}
H_0(N_{3-i},\ W_i) & = & \overline{L}_{3-i}(\lambda)\otimes_E\left(i^{L_{3-i}}_{B\cap L_{3-i}}(\chi^\infty_{s_{3-i}})\oplus\mathfrak{d}^\infty_{P_{3-i}}\right) \\
H_1(N_{3-i},\ W_i) & = & \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\left(i^{L_{3-i}}_{B\cap L_{3-i}}(\chi^\infty_{s_{3-i}})\oplus\mathfrak{d}^\infty_{P_{3-i}}\right)
\end{array}. \tag{4.25}$$

We notice that $Z(L_{3-i}(\mathbf{Q}_p))$ acts on $\overline{L}_{3-i}(\lambda)$ and $\overline{L}_{3-i}(s_i\cdot\lambda)$ (resp. $\mathfrak{d}^\infty_{P_{3-i}}$ and $\pi^\infty_{3-i,1}$) via different characters, and that $i^{L_{3-i}}_{B\cap L_{3-i}}(\chi^\infty_{s_{3-i}})$ has cosocle $1_{L_{3-i}}$. Hence we deduce from (4.25) the equalities

$$\begin{aligned}\mathrm{Ext}^1_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(H_0(N_{3-i},\ W_i),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\pi^\infty_{3-i,1}\right)&=0\\ \mathrm{Hom}_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(H_1(N_{3-i},\ W_i),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\pi^\infty_{3-i,1}\right)&=0\end{aligned}$$

which imply by Lemma 2.1 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^\infty_{3-i,1})\right)=0.$$

Hence we finish the proof of (4.23) by the embedding (4.24). $\square$

Lemma 4.8. *We have for each $i=1,2$:*

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{ --- } C_{s_i,s_i}\ ,\ C^2_{s_i,1}\right)&=0\\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}} \text{ --- } C^2_{s_i,s_is_{3-i}}\ ,\ C_{s_i,s_i}\right)&=0\\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \text{ --- } C^1_{s_i,s_is_{3-i}}\ ,\ C^1_{s_i,1}\right)&=0\\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\ C^2_{s_i,1}\ ,\ C^2_{s_i,s_is_{3-i}}\right)&=0\end{aligned}$$

Proof. We only prove that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{ --- } C_{s_i,s_i}\ ,\ C^2_{s_i,1}\right)=0 \tag{4.26}$$

as the other cases are similar. The surjection

$$\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2})\twoheadrightarrow\overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{ --- } C_{s_i,s_i}$$

and the embedding

$$C^2_{s_i,1}\hookrightarrow\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^\infty_{3-i,1})$$

induce an embedding

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{ --- } C_{s_i,s_i}\ ,\ C^2_{s_i,1}\right)\\ &\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2}),\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^\infty_{3-i,1})\right).\end{aligned} \tag{4.27}$$

It follows from the second paragraph of the list before Corollaire 5.2.1 of [Bre17] that

$$H_0(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2}))=\left(\overline{L}_{3-i}(\lambda)\oplus\overline{L}_{3-i}(s_i\cdot\lambda)\right)\otimes_E\pi^\infty_{3-i,2}$$

and

$$\begin{aligned}&H_1(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2}))\\ &\qquad=\left(\overline{L}_{3-i}(s_i\cdot\lambda)\oplus\overline{L}_{3-i}(s_is_{3-i}\cdot\lambda)\right)\otimes_E\pi^\infty_{3-i,2}\oplus I^{L_{3-i}}_{B\cap L_{3-i}}(\delta_{s_i\cdot\lambda})\\ &\qquad\qquad\oplus I^{L_{3-i}}_{B\cap L_{3-i}}\left(\delta_{s_i\cdot\lambda}\otimes_E\chi^\infty_{s_1s_2s_1}\right).\end{aligned}$$

We notice that $Z(L_{3-i}(\mathbf{Q}_p))$ acts on each direct summand of

$$H_k(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2}))$$

$(k=0,1)$ via a different character, and the only direct summand that produces the same character as $\overline{L}_{3-i}(s_i\cdot\lambda)\otimes\pi^\infty_{3-i,1}$ is $I^{L_{3-i}}_{B\cap L_{3-i}}(\delta_{s_i\cdot\lambda})$. However, we know that

$$\mathrm{cosoc}_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(I^{L_{3-i}}_{B\cap L_{3-i}}(\delta_{s_i\cdot\lambda})\right)=I^{L_{3-i}}_{B\cap L_{3-i}}\left(\delta_{s_{3-i}s_i\cdot\lambda}\right)$$

and thus

$$\mathrm{Hom}_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(I^{L_{3-i}}_{B\cap L_{3-i}}\left(\delta_{s_{3-i}s_i\cdot\lambda}\right),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes\pi^\infty_{3-i,1}\right)=0.$$

As a result, we deduce the equalities

$$\begin{aligned}
&\mathrm{Ext}^1_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(H_0(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2})),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\pi^\infty_{3-i,1}\right)=0\\
&\mathrm{Hom}_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(H_1(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2})),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\pi^\infty_{3-i,1}\right)=0
\end{aligned}$$

which imply by Lemma 2.1 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^\infty_{3-i,2}),\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^\infty_{3-i,1})\right)=0.$$

Hence we finish the proof of (4.26) by the embedding (4.27). □

Lemma 4.9. *Up to isomorphism, there exists a unique representation of the form*

$$\begin{tikzcd} & C^1_{s_{3-i}s_i,1} \arrow[dr, dash] & \\ C^2_{s_i,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{s_i,s_i} \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \arrow[ur, dash] & \end{tikzcd}$$

and a unique representation of the form

$$\begin{tikzcd} & C^1_{s_{3-i}s_i,s_{3-i}s_i} \arrow[dr, dash] & \\ C_{s_i,s_i} \arrow[ur, dash] \arrow[dr, dash] & & C^2_{s_i,s_is_{3-i}} \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}} \arrow[ur, dash] & \end{tikzcd}.$$

Proof. We only prove the first statement as the second one is similar. It follows from Proposition 4.4.2 of [Bre17] that there exists a unique representation of the form

$$\begin{tikzcd} & C^1_{s_{3-i}s_i,1} \arrow[dr, dash, dashed] & \\ C^2_{s_i,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{s_i,s_i} \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \arrow[ur, dash] & \end{tikzcd}$$

but it is not proven there whether its quotient

$$C^1_{s_{3-i}s_i,1}\ -\,-\,-\ C_{s_i,s_i} \tag{4.28}$$

is split or not. However, If (4.28) is split, then there exists a representation of the form

$$C^2_{s_i,1} \longrightarrow \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \longrightarrow C_{s_i,s_i}$$

which contradicts the first vanishing in Lemma 4.8, and thus we finish the proof. □

Remark 4.10. Our method used in Lemma 4.8 and in Lemma 4.9 is different from the one due to Y.Ding mentioned in part (ii) of Remark 4.4.3 of [Bre17]. It is not difficult to observe that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_i,s_i},\ \begin{tikzcd}[column sep=small, row sep=tiny] & C^1_{s_{3-i}s_i,1} \\ C^2_{s_i,1} \arrow[ur, dash] \arrow[dr, dash] & \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_i}\end{tikzcd}\right) = 1 \tag{4.29}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C^2_{s_i,s_is_{3-i}},\ \begin{tikzcd}[column sep=small, row sep=tiny] & C^1_{s_{3-i}s_i,s_{3-i}s_i} \\ C_{s_i,s_i} \arrow[ur, dash] \arrow[dr, dash] & \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}}\end{tikzcd}\right) = 1 \tag{4.30}$$

for each $i = 1, 2$. Similar methods as those used in Proposition 4.4.2 of [Bre17], in Lemma 4.8 and in Lemma 4.9 also imply the existence of a unique representation of the form

$$\begin{tikzcd}[column sep=small, row sep=tiny] & C_{s_{3-i}s_i,s_{3-i}} \arrow[dr, dash] & \\ C^1_{s_i,1} \arrow[ur, dash] \arrow[dr, dash] & & C^1_{s_i,s_is_{3-i}} \\ & \overline{L}(\lambda) \arrow[ur, dash] & \end{tikzcd}$$

or of the form

$$\begin{tikzcd}[column sep=small, row sep=tiny] & C_{s_{3-i}s_i,s_{3-i}} \arrow[dr, dash] & \\ C^2_{s_i,s_is_{3-i}} \arrow[ur, dash] \arrow[dr, dash] & & C^2_{s_i,1} \\ & \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \arrow[ur, dash] & \end{tikzcd}.$$

5. Computations of Ext II

In this section, we prove a few technical results which serve as a preparation to the construction and study of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ in Section 6. Note that we have defined the representation $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ in (2.22), which will be the starting point of the construction of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$. In order to add more and more Jordan–Hölder factors into $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ until we build up $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$, it is necessary for us to understand the extensions of various small length representations by certain subrepresentations of $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$. We compute the dimension of various such Ext-groups in this section, and a notable result is Proposition 5.4 which excludes the existence of certain representations of specific forms, using a key input from Proposition 3.5. A summary of different representations defined in this section can be found in Remark 5.10.

We recall the definition of $\Sigma_i(\lambda, \mathscr{L})$ for each $i = 1, 2$ and $\mathscr{L} \in E$ from the paragraph right before (2.21).

Lemma 5.1. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_i,s_i},\ \Sigma_i(\lambda, \mathscr{L}_i)\right) = 1$$

for each $i = 1, 2$.

Proof. We only prove that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_1,s_1},\ \Sigma_1(\lambda,\mathscr{L}_1)\right) = 1 \tag{5.1}$$

as the proof of the other equality is similar. We note that $\Sigma_1(\lambda,\mathscr{L}_1)$ admits a subrepresentation of the form

$$W \overset{\mathrm{def}}{=} \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \;\text{—}\; C^2_{s_1,1} \begin{array}{l} \diagup\ C^1_{s_2s_1,1} \\ \diagdown\ \overline{L}(\lambda)\otimes_E v^\infty_{P_1} \end{array}$$

due to Lemma 3.34, Lemma 3.37 and Remark 3.38 of [BD20]. Therefore $\Sigma_1(\lambda,\mathscr{L}_1))$ admits a separated and exhaustive filtration such that W appears as one term of the filtration and the only reducible graded piece is

$$V_1 \overset{\mathrm{def}}{=} C^2_{s_1,1} \begin{array}{l} \diagup\ C^1_{s_2s_1,1} \\ \diagdown\ \overline{L}(\lambda)\otimes_E v^\infty_{P_1} \end{array}.$$

It follows from Lemma 4.4.1 and Proposition 4.2.1 of [Bre17] as well as our Lemma 4.6 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_1,s_1},\ V\right) = 0 \tag{5.2}$$

for all graded pieces V different from V_1. On the other hand, we have

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_1,s_1},\ V_1\right) = 1 \tag{5.3}$$

due to (4.29) and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_1,s_1},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 0 \tag{5.4}$$

by Proposition 4.6.1 of [Bre17]. Hence we finish the proof by combining (5.2), (5.3), (5.4) and part (ii) of Proposition 2.2. □

We define $\Sigma_i^+(\lambda,\mathscr{L}_i)$ as the unique (up to isomorphism) non-split extension given by a non-zero element in

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_i,s_i},\ \Sigma_i(\lambda,\mathscr{L}_i)\right)$$

for each $i = 1,2$. Then we consider the amalgamate sum of $\Sigma_1^+(\lambda,\mathscr{L}_1)$ and $\Sigma_2^+(\lambda,\mathscr{L}_2)$ over $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ and denote it by $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$. In particular, $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ has the following form

$$\mathrm{St}_3^{\mathrm{an}}(\lambda) \begin{array}{l} \diagup\ v^{\mathrm{an}}_{P_1}(\lambda) \;\text{——}\; C_{s_1,s_1} \\ \diagdown\ v^{\mathrm{an}}_{P_2}(\lambda) \;\text{——}\; C_{s_2,s_2} \end{array}. \tag{5.5}$$

Lemma 5.2. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}},\ \Sigma_i^+(\lambda,\mathscr{L}_i)\right) = 3$$

for each $i = 1,2$.

Proof. By symmetry, it suffices to prove that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)=3.$$

This follows immediately from Lemma 3.42 of [Bre17] as our $\Sigma_1^+(\lambda,\mathscr{L}_1)$ can be identified with the locally analytic representation $\widetilde{\Pi}^1(\lambda,\psi)$ defined before (3.76) of [Bre17] up to changes on notation. □

Lemma 5.3. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

for each $i=1,2$.

Proof. The short exact sequence

$$\Sigma_2^+(\lambda,\mathscr{L}_2)\hookrightarrow \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow \left(\ v_{P_1}^{\mathrm{an}}(\lambda) \text{ —— } C_{s_1,s_1}\ \right)$$

induces the following long exact sequence

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda) \text{ —— } C_{s_1,s_1}\ \right)\\
&\qquad\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma_2^+(\lambda,\mathscr{L}_2)\right)\\
&\qquad\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\
&\qquad\qquad\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda) \text{ —— } C_{s_1,s_1}\ \right).
\end{aligned}$$

According to Proposition 4.1 and Lemma 4.4, we observe that

$$\dim_E \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda) \text{ —— } C_{s_1,s_1}\ \right)=1$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda) \text{ —— } C_{s_1,s_1}\ \right)=0$$

by a simple dévissage, which together with Lemma 5.2 and the long exact sequence above imply that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2.$$

The proof for

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

is parallel. □

Proposition 5.4. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_{3-i},\ \Sigma_i^+(\lambda,\mathscr{L}_i))=0$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_{3-i},\ \Sigma_i(\lambda,\mathscr{L}_i))=0$$

for each $i=1,2$.

Proof. It is clear that

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_{3-i},\ C_{s_i,s_i})=0,$$

which together with a simple dévissage give us an embedding

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_{3-i},\ \Sigma_i(\lambda,\mathscr{L}_i))\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_{3-i},\ \Sigma_i^+(\lambda,\mathscr{L}_i))$$

for each $i = 1, 2$. Without loss of generality, it suffices to show the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_2,\ \Sigma_1^+(\lambda, \mathscr{L}_1)) = 0. \tag{5.6}$$

We define $\nu \overset{\mathrm{def}}{=} \lambda_{T_2,\iota_{T,1}}$ (which is the restriction of λ from T to T_2 via the embedding $\iota_{T,1}\colon T_2 \hookrightarrow T$) and view $\Sigma^+_{\mathrm{GL}_2}(\nu, \mathscr{L}_1)$ (which is defined before Proposition 3.5) as a locally analytic representation of $L_1(\mathbf{Q}_p)$ via the projection $L_1(\mathbf{Q}_p) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Q}_p)$ and denote it by $\Sigma^+_{L_1}(\lambda, \mathscr{L}_1)$. We note by the definition of $\Sigma_1(\lambda, \mathscr{L}_1)$ (cf. Section 2.4) that we have an isomorphism

$$\Sigma_1(\lambda, \mathscr{L}_1) \xrightarrow{\sim} I^{\mathrm{GL}_3}_{P_1}(\Sigma_{L_1}(\lambda, \mathscr{L}_1)) \,/\, \left(v^{\mathrm{an}}_{P_2}(\lambda) \text{ --- } \overline{L}(\lambda) \right).$$

Upon viewing $\widetilde{I}(s \cdot \nu)$ as a locally analytic representation of $L_1(\mathbf{Q}_p)$ via the projection $L_1(\mathbf{Q}_p) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Q}_p)$, we deduce an isomorphism

$$C_{s_1,s_1} \cong \mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(I^{\mathrm{GL}_3}_{P_1}\left(\widetilde{I}(s\cdot\nu)\right)\right),$$

which together with the short exact sequence

$$\Sigma^+_{\mathrm{GL}_2}(\nu, \mathscr{L}_1) \hookrightarrow \Sigma^+_{\mathrm{GL}_2}(\nu, \mathscr{L}_1) \twoheadrightarrow \widetilde{I}(s \cdot \nu)$$

implies an injection

$$\Sigma_1^+(\lambda, \mathscr{L}_1) \hookrightarrow I^{\mathrm{GL}_3}_{P_1}\left(\Sigma^+_{L_1}(\lambda, \mathscr{L}_1)\right) \,/\, \left(v^{\mathrm{an}}_{P_2}(\lambda) \text{ --- } \overline{L}(\lambda) \right).$$

We use the shortened notation

$$V \overset{\mathrm{def}}{=} I^{\mathrm{GL}_3}_{P_1}\left(\Sigma^+_{L_1}(\lambda, \mathscr{L}_1)\right) \,/\, \left(v^{\mathrm{an}}_{P_2}(\lambda) \text{ --- } \overline{L}(\lambda) \right).$$

and obtain an injection (using a vanishing of Hom)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^+(\lambda, \mathscr{L}_1)\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ V\right). \tag{5.7}$$

We clearly have an exact sequence

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ I^{\mathrm{GL}_3}_{P_1}\left(\Sigma^+_{L_1}(\lambda, \mathscr{L}_1)\right)\right) \\ &\quad \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ V\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ v^{\mathrm{an}}_{P_2}(\lambda) \text{ --- } \overline{L}(\lambda)\right). \end{aligned} \tag{5.8}$$

We note that $W_2 \cong i^{\mathrm{GL}_3}_{P_1}(\mathfrak{d}^\infty_{P_1})$ (cf. (53) of [Bre17]). Then we deduce from (48) as well as the first paragraph of the list before Corollaire 5.2.1 of [Bre17] that

$$\begin{aligned} H_0(N_1,\ W_2) &= \overline{L}_1(\lambda) \otimes_E \left(i^{L_1}_{B\cap L_1}(\chi^\infty_{s_1}) \oplus \mathfrak{d}^\infty_{P_1}\right) \\ H_1(N_1,\ W_2) &= \overline{L}_1(s_2\cdot\lambda) \otimes_E \left(i^{L_1}_{B\cap L_1}(\chi^\infty_{s_1}) \oplus \mathfrak{d}^\infty_{P_1}\right). \end{aligned}$$

Hence we observe that

$$\mathrm{Hom}_{L_1(\mathbf{Q}_p),\lambda}\left(H_1(N_1,\ W_2),\ \Sigma^+_{L_1}(\lambda, \mathscr{L}_1)\right) = 0$$

from the action of $Z(L_1(\mathbf{Q}_p))$ and

$$\mathrm{Ext}^1_{L_1(\mathbf{Q}_p),\lambda}\left(H_0(N_1,\ W_2),\ \Sigma^+_{L_1}(\lambda, \mathscr{L}_1)\right) = 0$$

according to Proposition 3.5 and the natural identification

$$\mathrm{Ext}^1_{L_1(\mathbf{Q}_p),\lambda}(-,-) \cong \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(-,-).$$

As a result, we deduce

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ I^{\mathrm{GL}_3}_{P_1}\left(\Sigma^+_{L_1}(\lambda, \mathscr{L}_1)\right)\right) = 0 \tag{5.9}$$

from Lemma 2.1. We know that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \ v^{\mathrm{an}}_{P_2}(\lambda) \text{ --- } \overline{L}(\lambda)\ \right) = 0 \tag{5.10}$$

due to Proposition 4.1, Lemma 4.5 and a simple dévissage. Hence we finish the proof of (5.6) by combining (5.7), (5.8), (5.9) and (5.10). ☐

Lemma 5.5. *We have*

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+_i(\lambda,\mathscr{L}_i)\right) = 3 \tag{5.11}$$

for each $i=1,2$,

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = 2 \tag{5.12}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = 1. \tag{5.13}$$

Proof. We claim that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C_{s_i,s_i}\right) = \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C_{s_i,s_i}\right) = 0 \tag{5.14}$$

using Lemma 4.4 and Lemma 4.5. Hence the equalities (5.12) and (5.13) follow directly from Lemma 2.11 and (5.14), using a long exact sequence induced from the short exact sequence

$$\Sigma_i(\lambda,\mathscr{L}_i) \hookrightarrow \Sigma^+_i(\lambda,\mathscr{L}_i) \twoheadrightarrow C_{s_i,s_i}.$$

Due to a similar argument using (5.14), we only need to show that

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right) = 3 \tag{5.15}$$

to finish the proof of (5.11). The short exact sequence

$$\mathrm{St}^{\mathrm{an}}_3(\lambda) \hookrightarrow \Sigma_i(\lambda,\mathscr{L}_i) \twoheadrightarrow v^{\mathrm{an}}_{P_i}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right) \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right)\\
&\quad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}^{\mathrm{an}}_3(\lambda)\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right)\\
&\quad\quad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right).
\end{aligned} \tag{5.16}$$

We know that

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}^{\mathrm{an}}_3(\lambda)\right) = 5$$

by Lemma 2.7. It follows from Proposition 4.1, Lemma 4.4, Lemma 4.5 and a simple dévissage that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right) = 2 \tag{5.17}$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right) = 0. \tag{5.18}$$

In order to deduce (5.15) from (5.16), it remains to show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right) = 0. \tag{5.19}$$

The short exact sequence

$$\left(\ v^{\mathrm{an}}_{P_{3-i}}(\lambda) \text{ --- } \overline{L}(\lambda)\ \right) \hookrightarrow I^{\mathrm{GL}_3}_{P_i}\left(\Sigma_{L_i}(\lambda,\mathscr{L}_i)\right) \twoheadrightarrow \Sigma_i(\lambda,\mathscr{L}_i)$$

induces

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_{3-i}}(\lambda) — \overline{L}(\lambda)\right)$$
$$\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ I^{\mathrm{GL}_3}_{P_i}(\Sigma_{L_i}(\lambda,\mathscr{L}_i))\right) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right)$$

by the vanishing

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_{3-i}}(\lambda) — \overline{L}(\lambda)\right) = 0$$

using Proposition 4.1 and Lemma 4.5. Therefore we only need to show that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_{3-i}}(\lambda) — \overline{L}(\lambda)\right) = 1 \tag{5.20}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ I^{\mathrm{GL}_3}_{P_i}(\Sigma_{L_i}(\lambda,\mathscr{L}_i))\right) = 1. \tag{5.21}$$

The equality (5.21) follows from Lemma 2.1 and the facts

$$\dim_E \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(H_0(N_i,\ \overline{L}(\lambda)),\ \Sigma_{L_i}(\lambda,\mathscr{L}_i)\right) = 1,$$
$$\mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(H_1(N_i,\ \overline{L}(\lambda)),\ \Sigma_{L_i}(\lambda,\mathscr{L}_i)\right) = 0$$

where the first equality essentially follows from Lemma 3.14 of [BD20] and the second equality follows from checking the action of $Z(L_i(\mathbf{Q}_p))$. On the other hand, (5.20) follows from (5.17) and Proposition 4.1 by a simple dévissage. Hence we finish the proof. □

Proposition 5.6. *The short exact sequence*

$$\overline{L}(\lambda)\otimes_E v^{\infty}_{P_i} \hookrightarrow W_i \twoheadrightarrow \overline{L}(\lambda)$$

induces the following isomorphisms

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_{3-i}},\ \Sigma^+_i(\lambda,\mathscr{L}_i)\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+_i(\lambda,\mathscr{L}_i)\right) \tag{5.22}$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_{3-i}},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \tag{5.23}$$

for each $i = 1, 2$.

Proof. The vanishing from Proposition 5.4 implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_{3-i}},\ \Sigma^+_i(\lambda,\mathscr{L}_i)\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+_i(\lambda,\mathscr{L}_i)\right)$$

is an injection and hence an isomorphism as both spaces have dimension three according to Lemma 5.2 and Lemma 5.5. The proof of (5.23) is similar. We emphasize that both (5.22) and (5.23) can be interpreted as the isomorphism given by the cup product with the one dimensional space

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v^{\infty}_{P_{3-i}}\right).$$

□

We define

$$\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\stackrel{\mathrm{def}}{=}\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \text{ and } \Sigma_i^\flat(\lambda,\mathscr{L}_i) \stackrel{\mathrm{def}}{=}\Sigma_i(\lambda,\mathscr{L}_i)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \tag{5.24}$$

for each $i=1,2$.

Lemma 5.7. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=1.$$

Proof. We define $\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the subrepresentation of $\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ that fits into the following short exact sequence

$$\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow\left(C^1_{s_2,1}\oplus C^1_{s_1,1}\right), \tag{5.25}$$

(cf. (2.4) for the definition of $C^1_{s_2,1}$, $C^1_{s_1,1}$, $C^2_{s_2,1}$ and $C^2_{s_1,1}$) and then define $\Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the subrepresentation of $\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ that fits into

$$\Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \twoheadrightarrow\left(\left(\ C^2_{s_1,1} \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_1}\ \right)\oplus\left(\ C^2_{s_2,1} \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_2}\ \right)\right). \tag{5.26}$$

It follows from Lemma 4.4 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ V)=0$$

for each $V\in\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ and therefore

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0 \tag{5.27}$$

by part (i) of Proposition 2.2. On the other hand, we know from Lemma 4.4 and Lemma 4.7 that there is no representation of the form

$$C^2_{s_i,1} \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{ --- } \overline{L}(\lambda)$$

which implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ C^2_{s_i,1} \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_i}\ \right)=0 \tag{5.28}$$

for each $i=1,2$. Hence we deduce from (5.26), (5.27), (5.28) and Proposition 2.2 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0. \tag{5.29}$$

Therefore (5.25) induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C^1_{s_2,1}\oplus C^1_{s_1,1}\right). \tag{5.30}$$

Assume first that (5.30) is a surjection, then we can choose a representation V_0 represented by a non-zero element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ lying in the preimage of $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C^1_{s_2,1}\right)$ under (5.30). Note that there is a short exact sequence

$$\Sigma_1^\flat(\lambda,\mathscr{L}_1)\hookrightarrow\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow v^{\mathrm{an}}_{P_2}(\lambda).$$

We observe that $\overline{L}(\lambda)$ lies above neither $C^1_{s_1,1}$ nor $\overline{L}(\lambda)\otimes_E v^\infty_{P_2}$ inside V_0 by our definition and (5.28), and thus V_0 is mapped to zero under the map

$$f\colon \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_2}(\lambda)\right)$$

which means that V_0 comes from an element in

$$\mathrm{Ker}(f)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right)$$

and in particular

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right)\neq 0 \tag{5.31}$$

The short exact sequence

$$\overline{L}(\lambda)\otimes_E v^\infty_{P_2}\hookrightarrow W_2\twoheadrightarrow \overline{L}(\lambda)$$

induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right). \tag{5.32}$$

On the other hand, the short exact sequence

$$\overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\hookrightarrow \Sigma_1(\lambda,\mathscr{L}_1)\twoheadrightarrow \Sigma^\flat_1(\lambda,\mathscr{L}_1) \tag{5.33}$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\right)\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1(\lambda,\mathscr{L}_1)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right)\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\right)\end{aligned}$$

which implies

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1(\lambda,\mathscr{L}_1)\right)\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right) \tag{5.34}$$

as we have

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\right)=\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\right)=0$$

from Lemma 4.2. We combine Proposition 5.4, (5.32) and (5.34) and deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat_1(\lambda,\mathscr{L}_1)\right)=0$$

which contradicts (5.31). In all, we have thus shown that

$$\begin{aligned}&\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\qquad<\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C^1_{s_2,1}\oplus C^1_{s_1,1}\right)=2\end{aligned} \tag{5.35}$$

by combining Lemma 4.4. Finally, the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\right)=0$$

from Proposition 4.1 implies an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

which finishes the proof by combining Lemma 2.11 and (5.35). □

Lemma 5.8. *We have*

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2.$$

Proof. The short exact sequence

$$\Sigma_i^\flat(\lambda,\mathscr{L}_i)\hookrightarrow\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow v_{P_{3-i}}^{\mathrm{an}}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}(5.36)\quad &\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right)\\ &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma_i^\flat(\lambda,\mathscr{L}_i)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right).\end{aligned}$$

It is easy to observe that

$$\dim_E\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right)=1$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right)=0$$

from Proposition 4.1 and Lemma 4.4. We can actually observe from Lemma 4.4 that the only $V\in\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}(\Sigma_i^\flat(\lambda,\mathscr{L}_i))$ such that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ V\right)\neq 0$$

is $V=C^2_{s_{3-i},1}$ and

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ C^2_{s_{3-i},1}\right)=1.$$

Hence we deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma_i^\flat(\lambda,\mathscr{L}_i)\right)\leq 1$$

and therefore (using (5.36))

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0 \tag{5.37}$$

for each $i=1,2$. The short exact sequence

$$\overline{L}(\lambda)\otimes_E\left(v_{P_1}^\infty\oplus v_{P_2}^\infty\right)\hookrightarrow W_0\twoheadrightarrow\overline{L}(\lambda)$$

induces

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\qquad\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E\left(v_{P_1}^\infty\oplus v_{P_2}^\infty\right),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}$$

which implies

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \tag{5.38}$$

by (5.37). Finally, the short exact sequence (5.33) induces

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\quad\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)\end{aligned}$$

which finishes the proof by Lemma 5.7, (5.38), and the fact

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)=1$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)=0$$

coming from Lemma 4.3. □

Lemma 5.9. *We have the inequality*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \ v^{\mathrm{an}}_{P_i}(\lambda) \text{ --- } C_{s_i,s_i}\ \right)\leq 2$$

for each $i=1,2$.

Proof. We know that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_j},\ C^1_{s_i,1}\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_j},\ \overline{L}(\lambda)\otimes_E v^\infty_{P_i}\right)=0$$

for $i,j=1,2$ from Proposition 4.1 and Lemma 4.4, and thus

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_j},\ v^{\mathrm{an}}_{P_i}(\lambda)\right)=0$$

for $i,j=1,2$ which together with (5.17) implies that

$$\begin{aligned}(5.39)\quad \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ v^{\mathrm{an}}_{P_i}(\lambda)\right)&\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ v^{\mathrm{an}}_{P_i}(\lambda)\right)\\ \leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right)&-\dim_E \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i},\ v^{\mathrm{an}}_{P_i}(\lambda)\right)\\ &=2-1=1.\end{aligned}$$

We also note that we have

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C_{s_i,s_i}\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_i},\ C_{s_i,s_i}\right)=0$$

by Lemma 4.4, which implies

$$(5.40)\quad \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_i,s_i}\right)\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}},\ C_{s_i,s_i}\right)=1$$

where the last equality follows again from Lemma 4.4. We finish the proof by combining (5.39) and (5.40) with the inequality

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \ v^{\mathrm{an}}_{P_i}(\lambda) \text{ --- } C_{s_i,s_i}\ \right)\\ &\qquad\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ v^{\mathrm{an}}_{P_i}(\lambda)\right)+\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_i,s_i}\right).\end{aligned}$$

□

Remark 5.10. The representations that appear in this section can be summarized by the following diagram

$$\begin{tikzcd}\Sigma^\flat_i(\lambda,\mathscr{L}_i) \arrow[d, hook] & \Sigma(\lambda,\mathscr{L}_i) \arrow[l, two heads] \arrow[r, hook] \arrow[d, hook] & \Sigma^+(\lambda,\mathscr{L}_i) \arrow[d, hook]\\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2) & \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2) \arrow[l, two heads] \arrow[r, hook] & \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\end{tikzcd}$$

for each $i=1,2$. Note that the first (resp. second, resp. third) column is defined in (5.24) (resp. (2.22), resp. (5.5)).

6. The family $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$

6.1. **Construction of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$.** In this section, we finish our construction of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ (cf. the paragraph before (6.28)), using results from Section 5. A summary about the technique used in this section can be found in Remark 6.11.

Lemma 6.1. *We have the inequality*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\leq 3.$$

Proof. The short exact sequence

$$\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow(C_{s_1,s_1}\oplus C_{s_2,s_2})$$

induces the exact sequence

$$\begin{aligned}(6.1)\quad \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)&\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\oplus C_{s_2,s_2}\right).\end{aligned}$$

We know that

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\oplus C_{s_2,s_2}\right)\\ &=\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\right)+\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_2,s_2}\right)=1+1=2\end{aligned}$$

by Lemma 4.4 and Lemma 4.5. We also know that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

by Lemma 5.8, and thus we obtain the following inequality:

$$\begin{aligned}(6.2)\quad &\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\quad+\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\oplus C_{s_2,s_2}\right)=2+2=4.\end{aligned}$$

Assume on the contrary that

$$(6.3)\qquad \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=4.$$

The short exact sequence

$$\Sigma_1^+(\lambda,\mathscr{L}_1)\hookrightarrow\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow\left(\ v_{P_2}^{\mathrm{an}}(\lambda)\ \text{——}\ C_{s_2,s_2}\ \right)$$

induces a long exact sequence

$$\begin{aligned}(6.4)\quad \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)&\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \ v_{P_2}^{\mathrm{an}}(\lambda)\ \text{——}\ C_{s_2,s_2}\ \right)\end{aligned}$$

which implies

$$(6.5)\qquad \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\geq 2$$

by (6.3) and Lemma 5.9. We consider a separated and exhaustive filtration of $\Sigma_1^+(\lambda,\mathscr{L}_1)$ whose only reducible graded piece is

$$C^2_{s_1,1}\ \text{——}\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\ .$$

It follows from Proposition 4.1, Lemma 4.4 together with a simple dévissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_1},\ C^2_{s_1,1}-\overline{L}(\lambda)\otimes_E v^\infty_{P_1}\right)=0,$$

which together with Lemma 4.4 implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_1},\ V\right)=0$$

for all graded pieces $V\neq\overline{L}(\lambda)\otimes_E\mathrm{St}^\infty_3$ of the filtration above. Hence we deduce by part (ii) of Proposition 2.2 an isomorphism of one dimensional spaces

$$\begin{aligned}(6.6)\quad &\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_1},\ \overline{L}(\lambda)\otimes_E\mathrm{St}^\infty_3\right)\\ &\qquad\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_1},\ \Sigma^+_1(\lambda,\mathscr{L}_1)\right).\end{aligned}$$

Then the short exact sequence

$$\overline{L}(\lambda)\otimes_E v^\infty_{P_1}\hookrightarrow W_0\twoheadrightarrow W_2$$

induces a long exact sequence

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^+_1(\lambda,\mathscr{L}_1)\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+_1(\lambda,\mathscr{L}_1)\right)\\ &\qquad\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^\infty_{P_1},\ \Sigma^+_1(\lambda,\mathscr{L}_1)\right),\end{aligned}$$

which together with (6.5) and (6.6) implies that

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^+_1(\lambda,\mathscr{L}_1)\right)\geq 1.$$

This contradicts Proposition 5.4. Hence we finish the proof. □

Proposition 6.2. *We have*

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=3.$$

Proof. The short exact sequence

$$\overline{L}(\lambda)\otimes_E\left(v^\infty_{P_2}\oplus v^\infty_{P_1}\right)\hookrightarrow W_0\twoheadrightarrow\overline{L}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}(6.7)\quad &\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E\left(v^\infty_{P_2}\oplus v^\infty_{P_1}\right),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\qquad\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}$$

and thus we have

$$\begin{aligned}(6.8)\quad &\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\geq\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\quad+\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda)\otimes_E\left(v^\infty_{P_2}\oplus v^\infty_{P_1}\right),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\quad-\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))=1+4-2=3\end{aligned}$$

due to Lemma 5.3 and Lemma 5.5, which finishes the proof by a comparison with Lemma 6.1. □

We define $\Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the unique non-split extension of $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ by $\overline{L}(\lambda)$ (cf. Lemma 2.11) and then define $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the amalgamate sum of $\Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ and $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ over $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$. Hence $\Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ has the form

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & v_{P_1}^{\mathrm{an}}(\lambda) \arrow[dr, dash] & \\
\mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & \overline{L}(\lambda) \\
 & v_{P_2}^{\mathrm{an}}(\lambda) \arrow[ur, dash] &
\end{tikzcd}$$

and $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ has the form

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & C_{s_1,s_1} \\
 & v_{P_1}^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & \\
\mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & \overline{L}(\lambda) \\
 & v_{P_2}^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & C_{s_2,s_2}
\end{tikzcd} .$$

Then we set

$$\Sigma^{*,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \stackrel{\mathrm{def}}{=} \Sigma^*(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$$

for $*=\{+\},\{\sharp\}$ and $\{\sharp,+\}$.

Lemma 6.3. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0$$

and

$$\begin{aligned}(6.9)\qquad &\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\qquad = \dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2.\end{aligned}$$

Proof. According to (5.14) and a simple dévissage, it suffices to show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0$$

and

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2.$$

The desired results then follow from Lemma 2.11, the long exact sequence

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\
&\quad\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\right)\\
&\quad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\
&\quad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\right),\end{aligned}$$

and the equalities (cf. Proposition 4.1)

$$\dim_E \quad \begin{array}{rcl}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)) &=& 1\\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)) &=& 0\\ \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)) &=& 0\end{array} .$$

$\square$

Remark 6.4. It is not difficult to observe from the proof of Lemma 5.5 and that of Lemma 6.3 that the following diagram

$$\begin{tikzcd}
 & \Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \arrow[dr, hook] & \\
\Sigma^{(}\lambda,\mathscr{L}_1,\mathscr{L}_2) \arrow[ur, hook] \arrow[dr, hook] & & \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \\
 & \Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \arrow[ur, hook] &
\end{tikzcd}$$

induces isomorphisms between two dimensional E-vector spaces

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{*}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

for $*=\varnothing,\{+\},\{\sharp\}$ and $\{\sharp,+\}$.

Lemma 6.5. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=0 \tag{6.10}$$

and

$$\begin{aligned}&\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\qquad=\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\geq 1.\end{aligned}$$

Proof. According to (5.14) and a simple dévissage, it suffices to show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=0 \tag{6.11}$$

and

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\geq 1. \tag{6.12}$$

The equality (6.11) follows from Lemma 5.7, Proposition 4.1 and a long exact sequence induced from the short exact sequence

$$\Sigma^{\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow\overline{L}(\lambda).$$

The inequality (6.12) follows from Proposition 4.1, (6.11), Lemma 6.3 and the long exact sequence

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty})\\ &\quad\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\end{aligned} \tag{6.13}$$

as we have

$$\begin{aligned}&\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\qquad\geq\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\qquad\quad-\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty})=2-1=1.\end{aligned}$$

□

We use the shortened notation $\underline{\mathscr{L}}\overset{\text{def}}{=}(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_1',\mathscr{L}_2')$ for a tuple of four elements in E. We recall from Proposition 5.6 an isomorphism of two dimensional spaces

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\xrightarrow{\sim}\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \tag{6.14}$$

for each $i = 1, 2$. We emphasize that the isomorphism (6.14) can be naturally explained by the cup product map

(6.15)
$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_i},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\cup \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v^{\infty}_{P_i}\right)\\ \rightarrow \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}$$

where $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v^{\infty}_{P_i}\right)$ is one dimensional by Proposition 4.1. According to Lemma 2.11 and Remark 6.4, we may abuse the notation

$$\{\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p}),c_0\}$$

for a basis of $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^*(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ for each $* = \varnothing, \{+\}, \{\sharp\}$ and $\{\sharp, +\}$. In particular, the element

$$c_0 + \mathscr{L}\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$$

generates a line in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ for each $\mathscr{L}\in E$. We define $\Sigma^+_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}'_i)$ as the representation represent by the preimage of

$$c_0 + \mathscr{L}'_i\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$$

in

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_i},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

via (6.14), for each $i = 1, 2$. Then we define $\Sigma^+(\lambda,\underline{\mathscr{L}})$ as the amalgamate sum of $\Sigma^+_1(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}'_1)$ and $\Sigma^+_2(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}'_2)$ over $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$, and therefore $\Sigma^+(\lambda,\underline{\mathscr{L}})$ has the form

$$\begin{array}{ccccccc} & \diagup & v^{\mathrm{an}}_{P_1}(\lambda) & \text{——} & C_{s_1,s_1} & \text{—} & \overline{L}(\lambda)\otimes_E v^{\infty}_{P_2}\\ \mathrm{St}^{\mathrm{an}}_3(\lambda) & & & & & & \\ & \diagdown & v^{\mathrm{an}}_{P_2}(\lambda) & \text{——} & C_{s_2,s_2} & \text{—} & \overline{L}(\lambda)\otimes_E v^{\infty}_{P_1}\end{array}.$$

We define $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ as the amalgamate sum of $\Sigma^+(\lambda,\underline{\mathscr{L}})$ and $\Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ over $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$, and thus $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ has the form

$$\begin{array}{ccccccc} & & & \diagup & C_{s_1,s_1} & \text{—} & \overline{L}(\lambda)\otimes_E v^{\infty}_{P_2}\\ & \diagup & v^{\mathrm{an}}_{P_1}(\lambda) & \diagdown & & & \\ \mathrm{St}^{\mathrm{an}}_3(\lambda) & & & & \overline{L}(\lambda) & & \\ & \diagdown & v^{\mathrm{an}}_{P_2}(\lambda) & \diagup & & & \\ & & & \diagdown & C_{s_2,s_2} & \text{—} & \overline{L}(\lambda)\otimes_E v^{\infty}_{P_1}\end{array}.$$

We also need the quotients

$$\Sigma^{+,\flat}(\lambda,\underline{\mathscr{L}}) \stackrel{\mathrm{def}}{=} \Sigma^+(\lambda,\underline{\mathscr{L}})/\overline{L}(\lambda)\otimes_E \mathrm{St}^{\infty}_3,\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}}) \stackrel{\mathrm{def}}{=} \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})/\overline{L}(\lambda)\otimes_E \mathrm{St}^{\infty}_3.$$

Lemma 6.6. *We have the inequality*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)\leq 1.$$

Proof. The short exact sequence

$$\Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\twoheadrightarrow \overline{L}(\lambda)\otimes_E\left(v^{\infty}_{P_2}\oplus v^{\infty}_{P_1}\right)$$

induces an injection

(6.16)
$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\left(v^{\infty}_{P_2}\oplus v^{\infty}_{P_1}\right)\right)$$

by Lemma 6.5. Note that we have

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\left(v_{P_2}^\infty\oplus v_{P_1}^\infty\right)\right)=2$$

by Proposition 4.1. Assume first that (6.16) is a surjection, and thus we can choose a representation V_0 represented by a non-zero element lying in the preimage of $\overline{L}(\lambda)\otimes_E v_{P_2}^\infty$ under (6.16). We observe that the very existence of V_0 implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(W_2,\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\Big)\neq 0. \tag{6.17}$$

We define

$$\Sigma_i^{+,\flat}(\lambda,\mathscr{L}_i)\overset{\mathrm{def}}{=}\Sigma_i^{+}(\lambda,\mathscr{L}_i)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$$

and thus obtain an embedding

$$\Sigma_i^{+,\flat}(\lambda,\mathscr{L}_i)\hookrightarrow\Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$$

for each $i=1,2$. We notice that the quotient $\Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)$ fits into a short exact sequence

$$\Big(\ v_{P_2}^{\mathrm{an}}(\lambda) \text{ — } \overline{L}(\lambda)\ \Big)\hookrightarrow\Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\twoheadrightarrow C_{s_2,s_2}.$$

We observe that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ C_{s_2,s_2}\right)=0 \tag{6.18}$$

from Lemma 4.4 and part (i) of Proposition 2.2. It follows from Proposition 4.1, Lemma 4.4 and a simple dévissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ C^1_{s_1,1}\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(\overline{L}(\lambda),\ \ C^1_{s_1,1} \text{ — } \overline{L}(\lambda)\ \Big)=0. \tag{6.19}$$

Hence if

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(W_2,\ \ C^1_{s_1,1} \text{ — } \overline{L}(\lambda)\ \Big)\neq 0,$$

there must exist a representation of the form

$$C^1_{s_1,1} \text{ — } \overline{L}(\lambda) \text{ — } \overline{L}(\lambda)\otimes_E v_{P_2}^\infty$$

which contradicts (6.19) and Lemma 4.7. As a result, we have shown that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(W_2,\ \ C^1_{s_1,1} \text{ — } \overline{L}(\lambda)\ \Big)=0$$

which together with Proposition 4.1 and part (i) of Proposition 2.2 implies

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(W_2,\ \ v_{P_2}^{\mathrm{an}}(\lambda) \text{ — } \overline{L}(\lambda)\ \Big)=0. \tag{6.20}$$

Now we can deduce

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(W_2,\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\Big)=0 \tag{6.21}$$

from (6.18) and (6.20). We combine (6.21) with Proposition 5.4 and conclude that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(W_2,\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\Big)=0$$

which contradicts (6.17). Consequently, the injection (6.16) must be strict and we finish the proof. □

According to Lemma 6.5, the short exact sequence

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \hookrightarrow \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}}) \twoheadrightarrow \overline{L}(\lambda)\otimes_E (v_{P_2}^{\infty}\oplus v_{P_1}^{\infty})$$

induces a long exact sequence:

(6.22)

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E (v_{P_2}^{\infty}\oplus v_{P_1}^{\infty})\right)\\ \xrightarrow{f} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right).\end{aligned}$$

According to (6.10) and a long exact sequence induced from

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty} \hookrightarrow \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2) \twoheadrightarrow \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2),$$

we obtain a natural embedding

(6.23)

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) \hookrightarrow \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right).$$

Proposition 6.7. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})) = 1$$

and the image of f is not contained in the image of (6.23).

Proof. We use a shortened notation for the two dimensional space

$$M \stackrel{\mathrm{def}}{=} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E (v_{P_2}^{\infty}\oplus v_{P_1}^{\infty})\right).$$

We have the following commutative diagram

(6.24)

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right) \arrow[r, hook, "i"] \arrow[d, hook, "h"'] & M \arrow[r, "f"] \arrow[d, equal] & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \arrow[d, "k"'] \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right) \arrow[r, hook, "j"] & M \arrow[r, "g"] & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)
\end{tikzcd}$$

where the middle vertical map is just an equality. We know that h is injective by the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) = 0$$

and k has a one dimensional image by (6.13). Both i and j are injective due to (6.9) and (6.10). Therefore by a simple diagram chasing we have

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)\\ &\qquad = \dim_E M - \dim_E \mathrm{Im}(g) \geq \dim_E M - \dim_E \mathrm{Im}(k) = 2-1 = 1\end{aligned}$$

by Lemma 6.5, and therefore

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right) = 1$$

by Lemma 6.6. Moreover, the map g has a one dimensional image and hence $k\circ f$ has one dimensional image, meaning that the image of f has dimension one or two and is not contained in $\mathrm{Ker}(k)$ (which is exactly the image of (6.23)). We consider the restriction of f to the direct summand $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right)$ which together with (cf. Remark 6.4)

(6.25)

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \cong \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{+,\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

gives a map

(6.26)
$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right).$$

According to our definition of $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$, the image of (6.26) is indeed given by the line

$$E\left(c_0+\mathscr{L}_i'\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})\right). \tag{6.27}$$

It is clear that (6.27) is different from the image of (6.23) which is exactly the line $E\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$. □

Proposition 6.8. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right)\leq 1$$

and the equality holds if and only if $\mathscr{L}_1'=\mathscr{L}_2'=\mathscr{L}_3$ for a certain $\mathscr{L}_3\in E$.

Proof. The inequality follows directly from Proposition 6.7 and the fact that the morphism h in (6.24) is an embedding. It follows from (6.22) that the equality

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right)=1$$

holds if and only if the image of f is one dimensional. Then we notice from the proof of Proposition 6.7 that the image of

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right)$$

under f is (6.27), up to the isomorphism (6.25). Therefore the image of f is one dimensional if and only if the two lines (6.27) (for $i=1,2$) coincide, which means that

$$\mathscr{L}_1'=\mathscr{L}_2'=\mathscr{L}_3$$

for a certain $\mathscr{L}_3\in E$. □

We use the notation $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ for the representation $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ when

$$\underline{\mathscr{L}}=(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_3).$$

We define $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ as the unique representation (up to isomorphism) given by a non-zero element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)$ according to Proposition 6.8. Therefore our $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ has the following form

$$\begin{tikzcd}[arrows=dash]
 & & C_{s_1,s_1} \arrow[r] & \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty} \arrow[dr] & \\
 & v_{P_1}^{\mathrm{an}}(\lambda) \arrow[ur] \arrow[dr] \arrow[rrr, dashed, bend left=5] & & & \\
\mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur] \arrow[dr] & & \overline{L}(\lambda) & & \overline{L}(\lambda) \\
 & v_{P_2}^{\mathrm{an}}(\lambda) \arrow[ur] \arrow[dr] \arrow[urrr, dashed] & & & \\
 & & C_{s_2,s_2} \arrow[r] & \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty} \arrow[uur] &
\end{tikzcd} \tag{6.28}$$

It follows from Proposition 4.1, Proposition 6.8, the definition of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and a simple dévissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)=0. \tag{6.29}$$

Remark 6.9. The definition of the invariant $\mathscr{L}_3\in E$ of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ obviously depends on the choice of c_0, and hence on the choice of a branch of p-adic dilogarithm function which is D_0. This is similar to the definition of the invariants $\mathscr{L}_1,\mathscr{L}_2\in E$ which depends on the choice of a branch of p-adic logarithm function which is $\log_0$. Note that the choice of p-adic logarithm function naturally determines a choice of p-adic dilogarithm function.

The following result will be useful in the proof of Theorem 7.1.

Proposition 6.10. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = 2.$$

Moreover, if V is the locally analytic representation determined by a line

$$M_V \subsetneq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

satisfying the condition that M_V is different from the image of (6.23), *then there exists a unique $\mathscr{L}_3 \in E$ such that*

$$V \cong \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3).$$

Proof. The short exact sequence

$$\overline{L}(\lambda)\otimes_E\left(v^\infty_{P_1}\oplus v^\infty_{P_2}\right)\hookrightarrow W_0 \twoheadrightarrow \overline{L}(\lambda)$$

together with Lemma 6.3 induce a commutative diagram

(6.30)

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_0,\ V^+) \arrow[r, "g_1"] \arrow[d, "h_1"'] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^+\right) \arrow[r, two heads, "k_1"] \arrow[d, hook, "h_2"'] & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^+\right) \arrow[d, "h_3"'] \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_0,\ V^{\sharp,+}) \arrow[r, hook, "g_2"] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^{\sharp,+}\right) \arrow[r, "k_2"] & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{\sharp,+}\right)
\end{tikzcd}$$

where we use shortened notation V_i^{alg} for $\overline{L}(\lambda)\otimes_E v^\infty_{P_i}$, V^+ for $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ and $V^{\sharp,+}$ for $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ to save space. We observe that g_2 is an injection due to Lemma 6.3, k_1 is a surjection by the proof of Proposition 6.2, h_3 is an isomorphism by Proposition 4.1 and a simple dévissage, and finally h_2 is an injection (due to an obvious vanishing of Hom). Assume that h_2 is not surjective, then any representation given by a non-zero element in $\mathrm{Coker}(h_2)$ admits a quotient of the form

$$C^1_{s_i,1} \text{ --- } \overline{L}(\lambda) \text{ --- } V_i^{\mathrm{alg}} \tag{6.31}$$

for $i=1$ or 2 due to Lemma 4.4. However, it follows from Lemma 4.7 that there is no representation of the form (6.31), which implies that h_2 is indeed an isomorphism, and hence k_2 is surjective by a diagram chasing. Therefore we conclude that

$$\begin{aligned}
&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ V^{\sharp,+}\right)\\
&\quad= \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^{\sharp,+}\right) - \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{\sharp,+}\right)\\
&= \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^{+}\right) - \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{+}\right) = 4-2 = 2.
\end{aligned}$$

The final claim on the existence of a unique $\mathscr{L}_3$ follows from Proposition 6.8, our definition of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and the observation that the restriction of k_2 to the direct summand

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_i^{\mathrm{alg}},\ V^{\sharp,+}\right)$$

induces isomorphisms

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_i^{\mathrm{alg}},\ V^{\sharp,+}\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{\sharp,+}\right)$$

which can be interpreted as the cup product morphism with the one dimensional space

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V_i^{\mathrm{alg}}\right)$$

for each $i = 1, 2$. $\square$

Remark 6.11. We give a summary on main ideas behind various techniques used in Section 5 and Section 6.1. Our overall goal is to construct the representation $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ using $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ as one of the building blocks, but the tricky point is *what representation to add* during each step of the construction. It is not difficult to construct $\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ from $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ by adding C_{s_1,s_1}, C_{s_2,s_2} and $\overline{L}(\lambda)$, each with multiplicity one, then the gap between $\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ and $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ is the length three locally algebraic representation W_0. If one adds $\overline{L}(\lambda) \otimes_E v_{P_1}^{\infty}$ and $\overline{L}(\lambda) \otimes_E v_{P_2}^{\infty}$ first, one obtains $\Sigma^{\sharp,+}(\lambda, \underline{\mathscr{L}})$ which depends on four invariants. Then it is not always possible to add one extra $\overline{L}(\lambda)$ to $\Sigma^{\sharp,+}(\lambda, \underline{\mathscr{L}})$, as the exact sequence (6.22) really depends on the choice of $\underline{\mathscr{L}} = (\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4)$. Nevertheless, we may consider the quotient

$$\Sigma^{\sharp,+,\flat}(\lambda, \underline{\mathscr{L}}) = \Sigma^{\sharp,+}(\lambda, \underline{\mathscr{L}})/\overline{L}(\lambda) \otimes_E \mathrm{St}_3^{\infty}$$

which technically helps us determine exactly for which $\underline{\mathscr{L}}$ we can add the extra $\overline{L}(\lambda)$ (cf. Proposition 6.7 and Proposition 6.8). Having a local-global compatibility theorem in mind, we expect that: if $\Sigma^{\sharp,+}(\lambda, \underline{\mathscr{L}})$ embeds into any Hecke eigenspace, an extra $\overline{L}(\lambda)$ should also appear in the Hecke eigenspace. Consequently, instead of adding $\overline{L}(\lambda) \otimes_E v_{P_1}^{\infty}$ and $\overline{L}(\lambda) \otimes_E v_{P_2}^{\infty}$ first, we view W_0 as a whole and study the extension of W_0 by $\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ (cf. Proposition 6.2 and Proposition 6.10). This will be crucial in the proof of Theorem 7.1. A frequently used technique (cf. Lemma 5.7 and Proposition 6.7) is the following: given a certain $V \in \mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ which appears in our computation, if we cannot determine $\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\cdot, V)$ directly, we study $\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\cdot, V^{\flat})$ first (with $V^{\flat} \stackrel{\mathrm{def}}{=} V/\overline{L}(\lambda) \otimes_E \mathrm{St}_3^{\infty}$), and then make use of a long exact sequence induced from

$$\overline{L}(\lambda) \otimes_E \mathrm{St}_3^{\infty} \hookrightarrow V \twoheadrightarrow V^{\flat}.$$

The idea behind is that V might depend on choice of invariants but $V^{\flat}$ doesn't, which usually makes the computation (via various dévissage) of $\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\cdot, V^{\flat})$ simpler than that of $\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\cdot, V)$.

6.2. **Structure of** $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$**.** In this section, we further clarify the internal structure of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ in Proposition 6.12, (6.42) and (6.43). In particular, we want to describe all subrepresentations of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ whose cosocle is isomorphic to $\overline{L}(\lambda)$. The picture (6.28) certainly does not contain enough information on this. At the end of this section, we also introduce the representation $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ (cf. the paragraph before Remark 6.14), which is slightly bigger than $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$.

We define $\Sigma^{\min,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as the unique subrepresentation of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form

$$\begin{array}{lllll} & & v_{P_1}^{\mathrm{an}}(\lambda) \text{ ------ } C_{s_1,s_1} \text{ --- } \overline{L}(\lambda) \otimes_E v_{P_2}^{\infty} \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) & \begin{array}{l} / \\ \backslash \end{array} & \\ & & v_{P_2}^{\mathrm{an}}(\lambda) \text{ ------ } C_{s_2,s_2} \text{ --- } \overline{L}(\lambda) \otimes_E v_{P_1}^{\infty} \end{array},$$

which fits into the short exact sequence

$$\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\twoheadrightarrow\overline{L}(\lambda)^{\oplus 2}. \tag{6.32}$$

We also define $\Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ as the unique subrepresentation of $\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ of the form

$$\begin{tikzcd} & \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty} \arrow[r, dash] & C_{s_1,s_1} \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & \\ & \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty} \arrow[r, dash] & C_{s_2,s_2} \end{tikzcd},$$

which fits into the short exact sequence

$$\begin{aligned}\Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)&\hookrightarrow\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\\ &\twoheadrightarrow\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right)\oplus C^1_{s_2,1}\oplus C^1_{s_1,1}.\end{aligned} \tag{6.33}$$

The short exact sequence (6.32) induces a long exact sequence

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)^{\oplus 2}\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)\\ &\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)^{\oplus 2}\right)\end{aligned}$$

which easily implies that

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)=2$$

by Proposition 4.1 and (6.29). We consider a separated and exhaustive filtration on $\Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ whose only reducible graded pieces are

$$C^1_{s_i,1} - \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}$$

for $i=1,2$. According to Lemma 4.4 and Lemma 4.7, we deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)=0$$

for all graded pieces V of the filtration above, which implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)=0.$$

Therefore (6.33) induces an injection of a two dimensional space into a four dimensional space

$$\begin{aligned}&M^{\min}\overset{\mathrm{def}}{=}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)\\ &\hookrightarrow M^+\overset{\mathrm{def}}{=}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right)\oplus C^1_{s_2,1}\oplus C^1_{s_1,1}\right).\end{aligned} \tag{6.34}$$

It follows from the definition of $\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ that we have embeddings

$$\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

which allow us to identify

$$M^-\overset{\mathrm{def}}{=}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

with a line in $M^{\min}$. We use the number $1,2,3,4$ to index the four representations $\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}$, $\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}$, $C^1_{s_2,1}$ and $C^1_{s_1,1}$ respectively, and we use the notation M_I for each subset $I\subseteq\{1,2,3,4\}$ to denote the corresponding subspace of M^+ with dimension the cardinality of I. For example, $M_{\{1,2\}}$ denotes the two dimensional subspace

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right)\right)$$

of M^+.

Proposition 6.12. *We have the following characterizations of $M^{\min}$ inside M^+:*

$$M^{\min} \cap M_{\{i,j\}} = 0 \text{ for } \{i,j\} \neq \{3,4\},$$

$$M^{\min} \cap M_{\{1,3,4\}} = M^{\min} \cap M_{\{2,3,4\}} = M^{\min} \cap M_{\{3,4\}} = M^-,$$

and

$$M^{\min} = (M^{\min} \cap M_{\{1,2,3\}}) \oplus (M^{\min} \cap M_{\{1,2,4\}}).$$

Proof. As $C^1_{s_1,1}$ and $C^1_{s_2,1}$ are in the cosocle of $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$, it is immediate that

$$M^- \subseteq M_{\{3,4\}}.$$

It follows from (6.28) that

$$M^{\min} \not\subseteq M_{\{3,4\}}$$

and thus $M^{\min} \cap M_{\{3,4\}}$ is one dimensional which must coincide with M^-. The proof of Lemma 6.1 implies that $M^{\min} \not\subseteq M_{\{i,3,4\}}$ for each $i = 1, 2$ and therefore $M^{\min} \cap M_{\{i,3,4\}}$ is one dimensional, which together with the inclusion

$$M^{\min} \cap M_{\{3,4\}} \subseteq M^{\min} \cap M_{\{i,3,4\}}$$

for each $i = 1, 2$, implies that

$$M^{\min} \cap M_{\{i,3,4\}} = M^-.$$

We note from Proposition 5.4 that that

$$M^- \cap M_{\{3\}} = M^- \cap M_{\{4\}} = 0,$$

and thus

$$\begin{aligned} M^{\min} \cap M_{\{1,3\}} &= M^{\min} \cap (M_{\{1,3,4\}} \cap M_{\{1,3\}}) = (M^{\min} \cap M_{\{3,4\}}) \cap M_{\{1,3\}} \\ &= M^- \cap M_{\{3\}} = 0. \end{aligned}$$

Similarly, we conclude that

$$M^{\min} \cap M_{\{i,j\}} = M^- \cap M_{\{i,j\}} = 0$$

for each $\{i,j\} \neq \{3,4\}, \{1,2\}$. We define $\Sigma^{\min,-,\natural}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as the unique subrepresentation of $\Sigma^{\min,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ that fits into the short exact sequence

$$\Sigma^{\min,-,\natural}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \hookrightarrow \Sigma^{\min,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \twoheadrightarrow C^1_{s_1,1} \oplus C^1_{s_2,1} \oplus C_{s_1s_2s_1,1}$$

and then define

$$\Sigma^{\min,-,\natural,\flat}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \stackrel{\text{def}}{=} \Sigma^{\min,-,\natural}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty.$$

Assume for the moment that $M^{\min} \cap M_{\{1,2\}} \neq 0$, then we have

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-,\natural}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\right) \neq 0$$

which together with (cf. Proposition 4.1)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty\right) = 0$$

implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-,\natural,\flat}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\right) \neq 0. \tag{6.35}$$

We observe that there exists a direct sum decomposition

$$\Sigma^{\min,-,\natural,\flat}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = V_1 \oplus V_2$$

where V_i is a representation of the form

$$\begin{tikzcd}[column sep=small, row sep=tiny] & C^1_{s_{3-i}s_i,1} \arrow[r, dash] \arrow[rdd, dash] & C^2_{s_{3-i}s_i,1} & & \\ C^2_{s_i,1} \arrow[ru, dash] \arrow[rdd, dash] & & & \overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}} & . \\ & & C_{s_i,s_i} \arrow[ru, dash] & & \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \arrow[ru, dash] & & & \end{tikzcd}$$

Switching V_1 and V_2 if necessary, we can assume by (6.35) that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V_1\right) \neq 0. \tag{6.36}$$

We also have an embedding

$$V_1 \hookrightarrow\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_2}$$

which induces an embedding (using a vanishing of Hom)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V_1\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_2}\ \right)$$

which together with (6.36) implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_2}\ \right) \neq 0. \tag{6.37}$$

The short exact sequences

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \hookrightarrow \Sigma_1(\lambda,\mathscr{L}_1) \twoheadrightarrow \Sigma_1^\flat(\lambda,\mathscr{L}_1),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \hookrightarrow \Sigma_1^+(\lambda,\mathscr{L}_1) \twoheadrightarrow \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)$$

induce isomorphisms
(6.38)

$$\begin{array}{rcl} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1(\lambda,\mathscr{L}_1)\right) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) \\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right) \end{array}$$

by Lemma 4.2. Hence we deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right) = 0 \tag{6.39}$$

from Proposition 5.4 and (6.38). The surjection $W_2 \twoheadrightarrow \overline{L}(\lambda)$ induces an embedding (using a vanishing of Hom)

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right)$$

which together with (6.39) implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) = 0$$

and hence

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right) = 0 \tag{6.40}$$

by (5.14) and a simple dévissage. It follows from (6.39) and (6.40) that there does not exists a representation of the form

$$\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{ --- } \overline{L}(\lambda)\otimes_E v^\infty_{P_2} \text{ --- } \overline{L}(\lambda)$$

or of the form

$$\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{ --- } \overline{L}(\lambda)\ ,$$

and therefore

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{ --- } \overline{L}(\lambda)\otimes_E v_{P_2}^\infty\right)=0$$

which contradicts (6.37). A a result, we have shown that

$$M^{\min}\cap M_{\{1,2\}}=0.$$

As $M^-\not\subseteq M_{\{1,2,i\}}$ for $i=3,4$, we deduce that both $M^{\min}\cap M_{\{1,2,3\}}$ and $M^{\min}\cap M_{\{1,2,4\}}$ are one dimensional. On the other hand, since we know that

$$(M^{\min}\cap M_{\{1,2,3\}})\cap(M^{\min}\cap M_{\{1,2,4\}})=M^{\min}\cap M_{\{1,2\}}=0,$$

we deduce the following direct sum decomposition

$$M^{\min}=(M^{\min}\cap M_{\{1,2,3\}})\oplus(M^{\min}\cap M_{\{1,2,4\}}).$$

□

It follows from Proposition 6.12 that the two dimensional E-vector space $M^{\min}$ has three special lines inside, given by M^-, $M^{\min}\cap M_{\{1,2,3\}}$ and $M^{\min}\cap M_{\{1,2,4\}}$. We use the notation $\overline{L}(\lambda)^i$ for copy of $\overline{L}(\lambda)$ inside $\overline{L}(\lambda)^{\oplus 2}$ corresponding to the one dimensional space $M^{\min}\cap M_{\{1,2,i+2\}}$ inside $M^{\min}$, and therefore we have a surjection

$$\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\twoheadrightarrow\left(\ C^1_{s_2,1} \text{ --- } \overline{L}(\lambda)^1\ \right)\oplus\left(\ C^1_{s_1,1} \text{ --- } \overline{L}(\lambda)^2\ \right). \tag{6.41}$$

In other words, given a subrepresentation $V\subseteq\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ whose cosocle is isomorphic to $\overline{L}(\lambda)$, if the radical (minimal subrepresentation $\mathrm{rad}(V)\subseteq V$ such that $V/\mathrm{rad}(V)$ is semisimple) of V does not map surjectively to

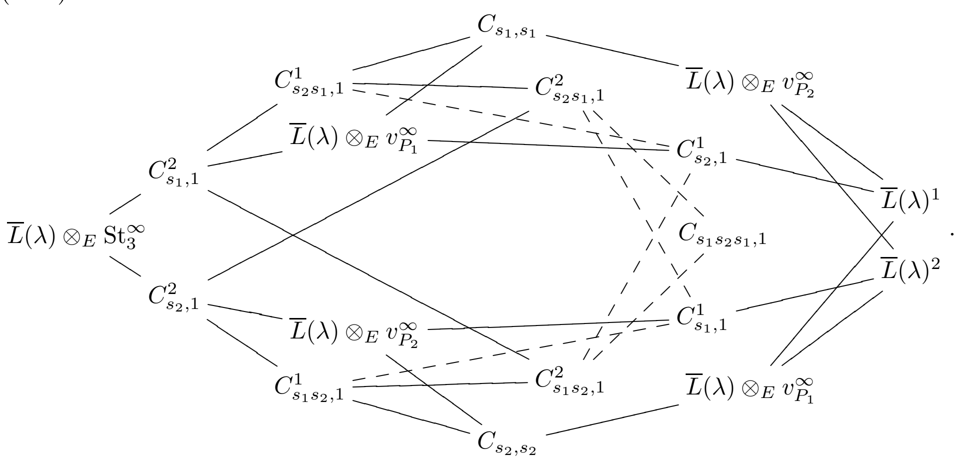
$$\left(\overline{L}(\lambda)\otimes_E v_{P_1}^\infty\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^\infty\right)\oplus C^1_{s_2,1}\oplus C^1_{s_1,1},$$

then V is either $\Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ (cf. M^-), or the unique subrepresentation of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ with cosocle $\overline{L}(\lambda)^i$ (cf. $M^{\min}\cap M_{\{1,2,i+2\}}$), for $i=1$ or 2.

According to our discussion above, the representation $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ has the following form:

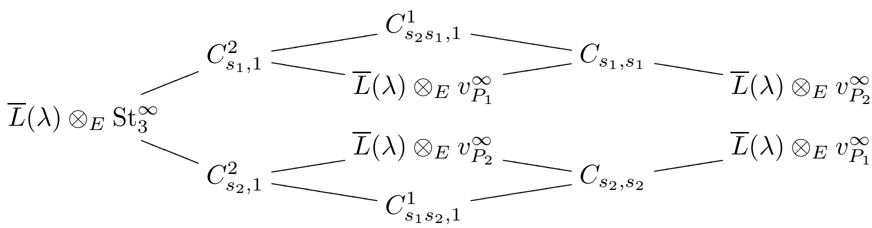
$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & C_{s_1,s_1} \arrow[r, dash] & \overline{L}(\lambda)\otimes_E v_{P_2}^\infty \arrow[dr, dash] \arrow[dddr, dash] & \\
 & v_{P_1}^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[rrr, dash] & & & \overline{L}(\lambda)^1 \\
\mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & & & \\
 & v_{P_2}^{\mathrm{an}}(\lambda) \arrow[dr, dash] \arrow[rrr, dash] & & & \overline{L}(\lambda)^2 \\
 & & C_{s_2,s_2} \arrow[r, dash] & \overline{L}(\lambda)\otimes_E v_{P_1}^\infty \arrow[ur, dash] \arrow[uuur, dash] &
\end{tikzcd} . \tag{6.42}$$

If we clarify the internal structure of $\mathrm{St}_3^{\mathrm{an}}(\lambda)$, $v_{P_1}^{\mathrm{an}}(\lambda)$ and $v_{P_2}^{\mathrm{an}}(\lambda)$ using Lemma 2.4, then $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ has the following form:
(6.43)

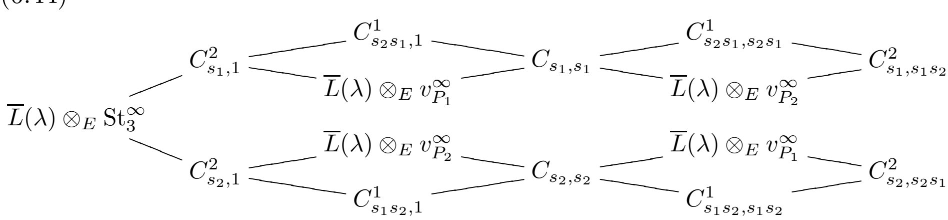


Remark 6.13. It is actually possible to show that all the possibly split extensions illustrated in (6.43) are non-split. However, the proof of these facts is quite technical and (6.43) is sufficient for our purpose (cf. Theorem 6.15 and Theorem 7.1), so we decide not to go further here.

We observe that $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ admits a unique subrepresentation $\Sigma^{\mathrm{Ext}^1,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form



which can be uniquely extend to a representation $\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form:
(6.44)



according to Section 4.4 and 4.6 of [Bre17] together with our Lemma 4.9. Finally, we define $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as the amalgamate sum of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ and $\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ over $\Sigma^{\mathrm{Ext}^1,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$.

Remark 6.14. It is actually possible to prove (by several technical computations of Ext-groups) that the quotient

$$\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty$$

and the quotient

$$\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty$$

are independent of the choices of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$.

6.3. **Relation to derived object.** In this section, as a byproduct of our construction in Section 6.1, we obtain an explicit complex (cf. Theorem 6.15) of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ that realizes the derived object constructed in Definition 5.19 of [Schr11]. We use a shortened notation $\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}$ for $\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\delta'_{Z,\lambda}}$, which is the abelian category of abstract modules over $D(\mathrm{GL}_3(\mathbf{Q}_p), E)$ with $D(Z, E)$ acting by $\delta'_{Z,\lambda}$ (cf. Section 2.1 and Section 2.3 for necessary notation). We define $\Sigma_i^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as the subrepresentation of $\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ (defined right after Proposition 6.8) that fits into the short exact sequence

$$\Sigma_i^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \hookrightarrow \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \twoheadrightarrow \overline{L}(\lambda) \otimes_E v_{P_i}^\infty$$

for each $i = 1, 2$. We use the notation $\mathcal{D}_i(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)'$ for the object in the derived category $\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)$ associated with the complex

$$\left[W'_{3-i} \longrightarrow \Sigma_i^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)'\right].$$

Theorem 6.15. *The object*

$$\mathcal{D}_i(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)' \in \mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)$$

fits into the distinguished triangle

$$\overline{L}(\lambda)' \longrightarrow \mathcal{D}_i(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)' \longrightarrow \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)'[-1] \xrightarrow{+1} \tag{6.45}$$

for each $i = 1, 2$. Moreover, the E-line inside

$$\begin{aligned} &\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) \\ &\qquad \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) \\ &\qquad \cong \mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)}\left(\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)'[-2],\ \overline{L}(\lambda)'\right) \end{aligned} \tag{6.46}$$

associated with the distinguished triangle (6.45) *is*

$$E(c_0 + \mathscr{L}_3 \kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p})). \tag{6.47}$$

In particular, for each $i = 1, 2$, $\mathcal{D}_i(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)'$ is isomorphic to the derived object constructed in Definition 5.19 of [Schr11] *(with Q there chosen to be zero) if $\mathscr{L}_1 = -\mathscr{L}$, $\mathscr{L}_2 = -\mathscr{L}'$ and $\mathscr{L}_3 = \mathscr{L}''$.*

Proof. It follows from Proposition 3.2 of [Schr11] that there is a unique (up to isomorphism) object

$$\mathcal{D}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)' \in \mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)$$

that fits into a distinguished triangle

$$\overline{L}(\lambda)' \longrightarrow \mathcal{D}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \longrightarrow \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)[-1] \xrightarrow{+1} \tag{6.48}$$

such that the element in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$ associated with (6.48) via (6.46) is (6.47). It follows from **TR2** (cf. Section 10.2.1 of [Wei94]) that

$$\mathcal{D}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)' \longrightarrow \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)'[-1] \longrightarrow \overline{L}(\lambda)'[1] \xrightarrow{+1} \tag{6.49}$$

is another distinguished triangle. The isomorphism (6.14) can be reinterpreted as the isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)}\left(\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1],\ \left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right)'\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)}\left(\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1],\ \overline{L}(\lambda)'[1]\right) \tag{6.50}$$

induced by the composition with

$$\mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)}\left(\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right)',\ \overline{L}(\lambda)'[1]\right).$$

As a result, each morphism

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1]\to\overline{L}(\lambda)'[1]$$

uniquely factors through a composition

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1]\to\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right)'\to\overline{L}(\lambda)'[1]$$

which induces a commutative diagram with four distinguished triangles
(6.51)

$$\begin{tikzcd}
 & & & {} & & \\
 & & \Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)' \arrow[ur, "+1"] \arrow[dr] & & {} & \\
 & \left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right)' \arrow[ur] \arrow[dr] & & \mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)' \arrow[ur, "+1"] \arrow[ddr] & & \\
 & & \overline{L}(\lambda)'[1] \arrow[ur] \arrow[drr] & & & \\
\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1] \arrow[uur] \arrow[urr] & & & & W'_{3-i}[1] \arrow[r, "+1"] \arrow[dr, "+1"] & {} \\
 & & & & & {}
\end{tikzcd}$$

by **TR4**. Hence we deduce that

$$\Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow W'_{3-i}[1]\xrightarrow{+1}$$

or equivalently

$$W'_{3-i}\longrightarrow\Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\xrightarrow{+1}$$

is a distinguished triangle. On the other hand, it is easy to see that $\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'$ fits into the distinguished triangle

$$W'_{3-i} \longrightarrow \Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)' \longrightarrow \mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)' \xrightarrow{+1}$$

and thus we conclude that

$$\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)' \cong \mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)' \in \mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)$$

by the uniqueness in Proposition 3.2 of [Schr11]. The last claim follows directly from (2.23) and an obvious comparison between our $\mathscr{L}_3$ and the $\mathscr{L}''$ in Definition 5.19 of [Schr11]. Hence we finish the proof. □

Remark 6.16. Now we explain the meaning of the notation $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$. The philosophy of p-adic local Langlands naturally predicts that one should be able to construct a family of locally analytic representations depending on three invariants, such that each representation in the family contains $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ as a subrepresentation. As a direct generalization of the case of $\mathrm{GL}_2(\mathbf{Q}_p)$, one firstly construct a family $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ that depends on two invariants $\mathscr{L}_1,\mathscr{L}_2 \in E$. It was firstly observed in [Schr11] that the third invariant should appear in

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \tag{6.52}$$

rather than $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$, purely due to the dimensional reason (cf. Lemma 2.11). In order to give a reasonable normalization of third invariant (in a way which conjecturally matches the third Fontaine–Mazur invariant on Galois side), one needs a special E-line inside (6.52). Then it turns out that the p-adic dilogarithm function admits a cohomological interpretation (cf. Section 5.3 of [Schr11]) which gives the required special E-line. Consequently, a family of abstract derived objects that depends on three invariants is constructed in Definition 5.19 of [Schr11]. Having the family of abstract derived objects in mind, our family $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ admits following characterization (cf. (6.42) and (6.43) for intuition): each representation in our family is minimal among representations V satisfying the following conditions

(i) V contains $\Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as a subrepresentation for some $\mathscr{L}_1,\mathscr{L}_1 \in E$;
(ii) there exists a complex with terms given by suitable subquotients of V, such that its associated object in $\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E),\lambda}\right)$ canonically determines a E-line in (6.52) of the form (6.47) for some $\mathscr{L}_3 \in E$.

7. Local-global compatibility

In this section, we prove our main result on local-global compatibility (cf. Theorem 7.1 and Corollary 7.5), which roughly says the following: up to suitable normalization and certain mild global assumption, if $\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}$ appears in the Hecke eigenspace associated with a global Galois representation, then there exists a unique choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3 \in E$ such that $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ also appears in the same Hecke eigenspace.

We are going to borrow most of the notation and assumptions from Section 6 of [Bre17]. We fix embeddings $\iota_\infty\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\iota_p\colon \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, an imaginary quadratic CM extension F of $\mathbf{Q}$ and a unitary group $G/\mathbf{Q}$ attached to the extension $F/\mathbf{Q}$ such that $G\times_{\mathbf{Q}} F \cong \mathrm{GL}_3$ and $G(\mathbf{R})$ is compact. If ℓ is a finite place of $\mathbf{Q}$ which splits completely in F, we have isomorphisms $\iota_{G,w}\colon G(\mathbf{Q}_\ell) \xrightarrow{\sim} G(F_w) \cong \mathrm{GL}_3(F_w)$

for each finite place w of F over ℓ. We assume that p splits completely in F, and we fix a finite place w_p of F dividing p and therefore $G(\mathbf{Q}_p) \cong G(F_{w_p}) \cong \mathrm{GL}_3(\mathbf{Q}_p)$.

We fix an open compact subgroup $U^p \subsetneq G(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$ of the form $U^p = \prod_{\ell \neq p} U_\ell$ where U_ℓ is an open compact subgroup of $G(\mathbf{Q}_\ell)$. Note that U^p is called *sufficiently small* if there exists $\ell \neq p$ such that U_ℓ has no non-trivial element with finite order. For each finite extension E of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}_p}$, we consider the following $\mathcal{O}_E$-lattice:

$$\widehat{S}(U^p, \mathcal{O}_E) \overset{\mathrm{def}}{=} \{f \colon G(\mathbf{Q})\backslash G(\mathbf{A}_{\mathbf{Q}}^{\infty})/U^p \to \mathcal{O}_E,\ f \text{ continuous}\} \tag{7.1}$$

inside the p-adic Banach space $\widehat{S}(U^p, E) \overset{\mathrm{def}}{=} \widehat{S}(U^p, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E$. The right translation of $G(\mathbf{Q}_p)$ on $G(\mathbf{Q})\backslash G(\mathbf{A}_{\mathbf{Q}}^{\infty})/U^p$ induces a p-adic continuous action of $G(\mathbf{Q}_p)$ on $\widehat{S}(U^p, \mathcal{O}_E)$ which makes $\widehat{S}(U^p, E)$ an admissible Banach representation of $G(\mathbf{Q}_p)$ in the sense of [ST02]. We use the notation $\widehat{S}(U^p, E)^{\mathrm{alg}} \subseteq \widehat{S}(U^p, E)^{\mathrm{an}}$ following Section 6 of [Bre17] for the subspaces of locally $\mathbf{Q}_p$-algebraic vectors and locally $\mathbf{Q}_p$-analytic vectors inside $\widehat{S}(U^p, E)$ respectively. Moreover, we have the following decomposition:

$$\widehat{S}(U^p, E)^{\mathrm{alg}} \otimes_E \overline{\mathbf{Q}_p} \cong \bigoplus_{\pi} (\pi_f^{v_0})^{U_p} \otimes_{\overline{\mathbf{Q}}} (\pi_{v_0} \otimes_{\overline{\mathbf{Q}}} W_p) \tag{7.2}$$

where the direct sum is over the automorphic representations π of $G(\mathbf{A}_{\mathbf{Q}})$ over $\mathbf{C}$ and W_p is the $\mathbf{Q}_p$-algebraic representation of $G(\mathbf{Q}_p)$ over $\overline{\mathbf{Q}_p}$ associated with the algebraic representation π_∞ of $G(\mathbf{R})$ over $\mathbf{C}$ via ι_p and ι_∞. In particular, each distinct π appears with multiplicity one (cf. the paragraph after (55) of [Bre17] for further references).

We use the notation $D(U^p)$ for the set of finite places ℓ of $\mathbf{Q}$ that are different from p, split completely in F and such that U_ℓ is a maximal open compact subgroup of $G(\mathbf{Q}_\ell)$. Then we consider the commutative polynomial algebra $\mathbf{T}(U^p) \overset{\mathrm{def}}{=} E[T_w^{(j)}]$ generated by the variables $T_w^{(j)}$ indexed by $j \in \{1, \cdots, n\}$ and w a finite place of F over a place ℓ of $\mathbf{Q}$ such that $\ell \in D(U^p)$. The algebra $\mathbf{T}(U^p)$ acts on $\widehat{S}(U^p, E)$, $\widehat{S}(U^p, E)^{\mathrm{alg}}$ and $\widehat{S}(U^p, E)^{\mathrm{an}}$ via the usual double coset operators. The action of $\mathbf{T}(U^p)$ commutes with that of $G(\mathbf{Q}_p)$.

We fix now $\alpha \in E^\times$, hence a Deligne–Fontaine module $\underline{D}$ over $\mathbf{Q}_p = F_{w_p}$ of rank three of the form

$$\underline{D} = Ee_2 \oplus Ee_1 \oplus Ee_0, \text{ with } \begin{cases} \varphi(e_2) &= \alpha e_2 \\ \varphi(e_1) &= p^{-1}\alpha e_1 \\ \varphi(e_0) &= p^{-2}\alpha e_0 \end{cases} \text{ and } \begin{cases} N(e_2) &= e_1 \\ N(e_1) &= e_0 \\ N(e_0) &= 0 \end{cases}. \tag{7.3}$$

and finally a tuple of Hodge–Tate weights $\underline{k} = (k_1 > k_2 > k_3)$. If $\rho \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(E)$ is an absolute irreducible continuous representation which is unramified at each finite place w lying over a finite place $\ell \in D(U^p)$, we can associate with ρ a maximal ideal $\mathfrak{m}_\rho \subseteq \mathbf{T}(U^p)$ with residual field E by the usual method described in the middle paragraph on Page 58 of [Bre17]. We use the notation $\star_{\mathfrak{m}_\rho}$ for spaces of localization and $\star[\mathfrak{m}_\rho]$ for torsion subspaces where $\star \in \{\widehat{S}(U^p, E), \widehat{S}(U^p, E)^{\mathrm{alg}}, \widehat{S}(U^p, E)^{\mathrm{an}}\}$.

We assume that there exists U^p and ρ such that

(i) ρ is absolutely irreducible and unramified at each finite place w of F over a place ℓ of $\mathbf{Q}$ satisfying $\ell \in D(U^p)$;

(ii) $\widehat{S}(U^p,E)^{\mathrm{alg}}[\mathfrak{m}_\rho]\neq 0$ (hence ρ is automorphic and $\rho_{w_p}\overset{\mathrm{def}}{=}\rho|_{\mathrm{Gal}(\overline{F_{w_p}}/F_{w_p})}$ is potentially semi-stable, cf. [BLGGT14], [Ca14]);

(iii) ρ_{w_p} has Hodge–Tate weights $\underline{k}$ and gives the Deligne–Fontaine module $\underline{D}$.

By identifying $\widehat{S}(U^p,E)^{\mathrm{alg}}$ with a representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ via ι_{G,w_p}, we have the following isomorphism up to normalization from (7.2) and [Ca14]:

$$\widehat{S}(U^{v_0},E)^{\mathrm{alg}}[\mathfrak{m}_\rho]\cong\left(\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det\right)^{\oplus d(U^p,\rho)} \tag{7.4}$$

for all (U^p,ρ) satisfying the conditions (i), (ii) and (iii), where $\lambda=(\lambda_1,\lambda_2,\lambda_3)=(k_1-2,k_2-1,k_3)$ and $d(U^p,\rho)\geq 1$ is an integer depending only on U^p and ρ.

Theorem 7.1. *We consider $U^p=\prod_{\ell\neq p}U_\ell$ and $\rho\colon\mathrm{Gal}(\overline{F}/F)\to\mathrm{GL}_3(E)$ such that*

(i) *ρ is absolutely irreducible and unramified at each finite place w of F lying above $D(U^p)$;*

(ii) *$\widehat{S}(U^p,E)^{\mathrm{alg}}[\mathfrak{m}_\rho]\neq 0$;*

(iii) *ρ has Hodge–Tate weights $\underline{k}$ and gives the Deligne–Fontaine module $\underline{D}$ as in* (7.3)*;*

(iv) *the Hodge filtration on $\underline{D}$ is non-critical in the sense of* (ii) *of Remark 6.1.4 of* [Bre17]*;*

(v) *only one automorphic representation π contributes to $\widehat{S}(U^p,E)^{\mathrm{alg}}[\mathfrak{m}_\rho]$.*

Then there exists a unique choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ such that:

(7.5)
$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right)\\&\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right).\end{aligned}$$

We recall several useful results from [Bre17] and [BH18]. We recall the upper-triangular Borel $\overline{B}$ as well as its radical $\overline{N}$ from Section 2.3 and let Π be an arbitrary admissible locally analytic representation of $\mathrm{GL}_3(\mathbf{Q}_p)$. We consider the subspace $\Pi[\overline{\mathfrak{n}}=0]\subseteq\Pi$ consisting of vectors killed by $\overline{\mathfrak{n}}$, and notice that $\Pi[\overline{\mathfrak{n}}=0]$ is stable under the action of $\overline{B}(\mathbf{Q}_p)$ and the smooth action of $\overline{N}(\mathbf{Q}_p)$. Hence the subspace of $\overline{N}(\mathbf{Z}_p)$-invariant $\Pi^{\overline{N}(\mathbf{Z}_p)}\subseteq\Pi[\overline{\mathfrak{n}}=0]$ is stable under the action of $\overline{B}(\mathbf{Z}_p)$ and $\mathfrak{t}$. For each character $\eta\colon U(\mathfrak{t})\to E$, we write $\Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]\subseteq\Pi^{\overline{N}(\mathbb{Z}_p)}$ for the subspace where $U(\overline{\mathfrak{b}})$ acts by η via $U(\overline{\mathfrak{b}})\twoheadrightarrow U(\mathfrak{t})$. We note that $\Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]=\Pi[\overline{\mathfrak{n}}=0][\mathfrak{t}=\eta]^{\overline{N}(\mathbb{Z}_p)}$ is stable under the action of $T(\mathbf{Q}_p)^+$ where

$$T(\mathbf{Q}_p)^+\overset{\mathrm{def}}{=}\{t\in T(\mathbf{Q}_p)\mid t\overline{N}(\mathbb{Z}_p)t^{-1}\subseteq\overline{N}(\mathbb{Z}_p)\}.$$

For each character $\chi\colon T(\mathbf{Q}_p)^+\to E^\times$, we write $\Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]_\chi\subseteq\Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]$ for the generalized eigenspace associated with χ.

Proposition 7.2. *Suppose that $U^p=\prod_{\ell\neq p}U_\ell$ is a sufficiently small open compact subgroup of $G(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$, $\widehat{S}(U^p,E)^{\mathrm{an}}\hookrightarrow\Pi\twoheadrightarrow\Pi_1$ is a short exact sequence inside $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$, $\chi\colon T(\mathbf{Q}_p)\to E^\times$ is a locally analytic character and $\eta\colon U(\mathfrak{t})\to E$ its derived character, then we have the following $T(\mathbf{Q}_p)^+$-equivariant short exact sequences of finite dimensional E-vector spaces*

$$(\widehat{S}(U^p,E)^{\mathrm{an}})^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]\hookrightarrow\Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]\twoheadrightarrow\Pi_1^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]$$

and

$$(\widehat{S}(U^p,E)^{\mathrm{an}})^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]_\chi \hookrightarrow \Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]_\chi \twoheadrightarrow \Pi_1^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t}=\eta]_\chi.$$

Proof. This is Proposition 6.3.3 of [Bre17] and Proposition 4.1 of [BH18]. □

Proposition 7.3. *We fix U^p and ρ as in Theorem 7.1. For a locally analytic character $\chi\colon T(\mathbf{Q}_p)\to E^\times$, we have*

$$\mathrm{Hom}_{T(\mathbf{Q}_p)^+}\Big(\chi\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ (\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho])^{\overline{N}(\mathbb{Z}_p)}\Big)\neq 0$$

if and only if $\chi=\delta_{T,\lambda}$.

Proof. This is Proposition 6.3.4 of [Bre17]. □

We recall the notation $i_B^{\mathrm{GL}_3}(\chi_w^\infty)$ for a smooth principal series for each $w\in W$ from Section 2.3. Given three locally analytic representations V_i for $i=1,2,3$ and two surjections $V_1\twoheadrightarrow V_2$ and $V_3\twoheadrightarrow V_2$, we use the notation $V_1\times_{V_2}V_3$ for the fiber product of V_1 and V_3 over V_2 with natural surjections $V_1\times_{V_2}V_3\twoheadrightarrow V_1$ and $V_1\times_{V_2}V_3\twoheadrightarrow V_3$. We also use the shortened notation V^{alg} for the maximally locally algebraic subrepresentation (given by the set of locally algebraic vectors) of a locally analytic representation V. We recall the set Ω (consisting of irreducible representations) from (2.6) and the sentence before it.

Proposition 7.4. *We fix U^p and ρ as in Theorem 7.1 and assume moreover that U^p is a sufficiently small open compact subgroup of $G(\mathbf{A}_{\mathbf{Q}}^{\infty,p})$. We also fix a non-split short exact sequence $V_1\hookrightarrow V_2\twoheadrightarrow V_3$ inside $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ such that $V_1\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det$ embeds into $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$. We conclude that:*

(i) *if $V_3\in\Omega$ is not locally algebraic, then we have an embedding*

$$V_2\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det\hookrightarrow\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho];$$

(ii) *if there exists a surjection*

$$\overline{L}(\lambda)\otimes_E i_B^{\mathrm{GL}_3}(\chi_w^\infty)\twoheadrightarrow V_3$$

for a certain $w\in W_{\mathrm{GL}_3}$, then there exists a quotient V_4 of

$$V_2\times_{V_3}\Big(\overline{L}(\lambda)\otimes_E i_B^{\mathrm{GL}_3}(\chi_w^\infty)\Big)$$

satisfying

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V_4)=V_4^{\mathrm{alg}}=\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty$$

such that we have an embedding

$$V_4\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det\hookrightarrow\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho].$$

Proof. This is an immediate generalization (or rather summary) of Section 6.4 of [Bre17]. More precisely, part (i) (resp. (ii)) generalizes the Étape 1 (resp. the Étape 2) of Section 6.4 of [Bre17]. □

Proof of Theorem 7.1. According to the Étape 1 and 2 of Section 6.2 of [Bre17], we may assume without loss of generality that U^p is sufficiently small and it is sufficient to show that there exists a unique choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ such that

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\Big(\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\Big)\neq 0. \tag{7.6}$$

For each $i=1,2$, we recall the representation $\Pi^i(\underline{k},\underline{D})$ constructed in Section 4.5 of [Bre17], which has the following form

(7.7)
$$\begin{tikzcd}[column sep=small, row sep=tiny] & C^1_{s_is_{3-i},1} \arrow[dr, dash] & & C^1_{s_is_{3-i},s_is_{3-i}} \arrow[dr, dash] & \\ C^2_{s_{3-i},1} \arrow[ur, dash] \arrow[dr, dash] & & C_{s_{3-i},s_{3-i}} \arrow[ur, dash] \arrow[dr, dash] & & C^2_{s_{3-i},s_{3-i}s_i} \\ & \overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}} \arrow[ur, dash] & & \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \arrow[ur, dash] & \end{tikzcd}$$

under notation (cf. Section 2.3) of our paper. We deduce from (7.7), (6.44) as well as the definition of $\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ before Remark 6.9 that $\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ contains a unique subrepresentation $\Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ of the form

$$\begin{tikzcd}[row sep=small] & \Pi^1(\underline{k},\underline{D}) \\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \arrow[ur, dash] \arrow[dr, dash] & \\ & \Pi^2(\underline{k},\underline{D}) \end{tikzcd} . \tag{7.8}$$

Moreover, $\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ is uniquely determined by $\Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ up to isomorphism. It is known by Étape 3 of Section 6.2 of [Bre17] that there is at most one choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ such that

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\Big(\Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\Big)\neq 0,$$

and thus there is at most one choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ such that (7.6) holds. As a result, it remains to show the existence of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ that satisfies (7.6). We notice that $\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ admits an increasing, separated and exhaustive filtration $\mathrm{Fil}_\bullet$ satisfying the following conditions

(i) the representations $\Sigma^{\mathrm{min}}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ (cf. their definition after Proposition 6.2 and Proposition 6.8) appear as two consecutive terms of the filtration;

(ii) each graded piece is either locally algebraic or irreducible.

As a result, the only reducible graded pieces of this filtration is the quotient

$$\Sigma^{\mathrm{min}}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)/\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\cong W_0.$$

Then we can prove the existence of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ satisfying (7.6) by reducing to the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\Big(&\mathrm{Fil}_{k+1}\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\\ &\qquad\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\Big)\\ \xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\Big(&\mathrm{Fil}_{k}\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\\ &\qquad\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\Big)\end{aligned} \tag{7.9}$$

for each $k\in\mathbb{Z}$. If

$$\mathrm{Gr}_k\overset{\mathrm{def}}{=}\mathrm{Fil}_{k+1}\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)/\mathrm{Fil}_k\Sigma^{\mathrm{min},+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

is not locally algebraic, then (7.9) is true by part (i) of Proposition 7.4. The only locally algebraic graded pieces of the filtration except $\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty$ are $\overline{L}(\lambda)\otimes_E v^\infty_{P_1}$, $\overline{L}(\lambda)\otimes_E v^\infty_{P_2}$ and W_0. The isomorphism (7.9) when the graded piece Gr_k equals

$\overline{L}(\lambda)\otimes_E v_{P_1}^\infty$ or $\overline{L}(\lambda)\otimes_E v_{P_2}^\infty$ has been treated in Étape 2 of Section 6.4 of [Bre17]. As a result, it remains to show that

(7.10)
$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right)\\ &\quad\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det,\ \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right)\end{aligned}$$

to finish the proof of Theorem 7.1. It follows from (53) of [Bre17] that $i_B^{\mathrm{GL}_3}(\chi_{s_1s_2s_1}^\infty)$ has the form

$$\begin{tikzcd}[column sep=small, row sep=tiny] & v_{P_1}^\infty \arrow[dr, dash] & \\ \mathrm{St}_3^\infty \arrow[ur, dash] \arrow[dr, dash] & & 1_3 \\ & v_{P_2}^\infty \arrow[ur, dash] & \end{tikzcd}$$

and thus there is a surjection

$$\overline{L}(\lambda)\otimes_E i_B^{\mathrm{GL}_3}(\chi_{s_1s_2s_1}^\infty)\twoheadrightarrow W_0.$$

According to part (ii) of Proposition 7.4, we only need to show that any quotient V of

$$V^\diamond\overset{\mathrm{def}}{=}\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\times_{W_0}\left(\overline{L}(\lambda)\otimes_E i_B^{\mathrm{GL}_3}(\chi_{s_1s_2s_1}^\infty)\right)$$

satisfying

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V)=V^{\mathrm{alg}}=\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty \tag{7.11}$$

must have the form

$$\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3')$$

for certain $\mathscr{L}_3'\in E$. We recall from Proposition 6.8 and our definition of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ afterwards that $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ fits into a short exact sequence

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\twoheadrightarrow W_0 \tag{7.12}$$

and thus $V^\diamond$ fits (by definition of fiber product) into a short exact sequence

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow V^\diamond\twoheadrightarrow i_B^{\mathrm{GL}_3}(\chi_{s_1s_2s_1}^\infty) \tag{7.13}$$

and in particular

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V^\diamond)=\left(\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)^{\oplus 2}.$$

Hence the condition (7.11) implies that V fits into a short exact sequence

$$\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\xrightarrow{j}V^\diamond\twoheadrightarrow V$$

and that

$$j\left(\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)\cap\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)=0\subseteq V^\diamond$$

which induces an injection

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow V.$$

Therefore V fits into a short exact sequence

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow V\twoheadrightarrow W_0$$

and thus corresponds to a line M_V inside

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

which is two dimensional by Proposition 6.10. Moreover, the condition (7.11) implies that M_V is different from the line given by the image of

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right).$$

Hence it follows from Proposition 6.10 that there exists $\mathscr{L}_3'\in E$ such that

$$V\cong \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3').$$

□

Corollary 7.5. *If a locally analytic representation* Π *of the form* (7.8) *is contained in* $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ *for a certain* U^p *and* ρ *as in Theorem* 7.1*, then there exists* $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ *uniquely determined by* Π *such that*

$$\Pi\hookrightarrow \Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3).$$

Proof. We fix U^p and ρ such that the embedding

$$\Pi\hookrightarrow \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho] \tag{7.14}$$

exists. Then (7.14) restricts to an embedding

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \hookrightarrow \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$$

which extends to an embedding

$$\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho] \tag{7.15}$$

for a unique choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ according to Theorem 7.1. The embedding (7.15) induces by restriction an embedding

$$\Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$$

and therefore we have

$$\Pi\cong \Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

by Théorème 6.2.1 of [Bre17]. In particular, we deduce an embedding

$$\Pi\hookrightarrow \Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

for certain invariants $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ determined by Π. □

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References

[Bre04] Christophe Breuil, *Invariant $\mathcal{L}$ et série spéciale p-adique* (French, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 4, 559–610, DOI 10.1016/j.ansens.2004.02.001. MR2097893

[Bre16] Christophe Breuil, *Socle localement analytique I* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **66** (2016), no. 2, 633–685. MR3477886

[Bre17] Christophe Breuil, Ext^1 *localement analytique et compatibilité local-global* (French, with French summary), Amer. J. Math. **141** (2019), no. 3, 611–703, DOI 10.1353/ajm.2019.0020. MR3956517

[BD20] Christophe Breuil and Yiwen Ding, *Higher $\mathcal{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$ and local-global compatibility*, Camb. J. Math. **8** (2020), no. 4, 775–951. MR4220371

[BH18] Christophe Breuil and Florian Herzig, *Towards the finite slope part for GL_n*, Int. Math. Res. Not. IMRN **24** (2020), 10495–10552, DOI 10.1093/imrn/rnz053. MR4190408

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Local-global compatibility for $l = p$, II* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, 165–179, DOI 10.24033/asens.2212. MR3205603

[Ca14] Ana Caraiani, *Monodromy and local-global compatibility for $l = p$*, Algebra Number Theory **8** (2014), no. 7, 1597–1646, DOI 10.2140/ant.2014.8.1597. MR3272276

[Cole82] Robert F. Coleman, *Dilogarithms, regulators and p-adic L-functions*, Invent. Math. **69** (1982), no. 2, 171–208, DOI 10.1007/BF01399500. MR674400

[Col10] Pierre Colmez, *Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ,Γ)-modules* (French, with English and French summaries), Astérisque **330** (2010), 281–509. MR2642409

[Ding19] Yiwen Ding, *Simple $\mathcal{L}$-invariants for GL_n*, Trans. Amer. Math. Soc. **372** (2019), no. 11, 7993–8042, DOI 10.1090/tran/7859. MR4029688

[Eme06] Matthew Emerton, *Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 5, 775–839, DOI 10.1016/j.ansens.2006.08.001. MR2292633

[Eme] Matthew Emerton, *A local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$: Special Issue: In honor of John H. Coates*, Pure Appl. Math. Q. **2** (2006), no. 2, 279–393, DOI 10.4310/PAMQ.2006.v2.n2.a1. MR2251474

[Hum08] James E. Humphreys, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008. MR2428237

[Koh07] Jan Kohlhaase, *Invariant distributions on p-adic analytic groups*, Duke Math. J. **137** (2007), no. 1, 19–62, DOI 10.1215/S0012-7094-07-13712-8. MR2309143

[Koh11] Jan Kohlhaase, *The cohomology of locally analytic representations*, J. Reine Angew. Math. **651** (2011), 187–240, DOI 10.1515/CRELLE.2011.013. MR2774315

[Or05] Sascha Orlik, *On extensions of generalized Steinberg representations*, J. Algebra **293** (2005), no. 2, 611–630, DOI 10.1016/j.jalgebra.2005.03.028. MR2173717

[OS15] Sascha Orlik and Matthias Strauch, *On Jordan-Hölder series of some locally analytic representations*, J. Amer. Math. Soc. **28** (2015), no. 1, 99–157, DOI 10.1090/S0894-0347-2014-00803-1. MR3264764

[OSc14] Sascha Orlik and Benjamin Schraen, *The Jordan-Hölder series of the locally analytic Steinberg representation*, Doc. Math. **19** (2014), 647–671. MR3247798

[ST02] P. Schneider and J. Teitelbaum, *Banach space representations and Iwasawa theory*, Israel J. Math. **127** (2002), 359–380, DOI 10.1007/BF02784538. MR1900706

[ST03] Peter Schneider and Jeremy Teitelbaum, *Algebras of p-adic distributions and admissible representations*, Invent. Math. **153** (2003), no. 1, 145–196, DOI 10.1007/s00222-002-0284-1. MR1990669

[ST05] Peter Schneider and Jeremy Teitelbaum, *Duality for admissible locally analytic representations*, Represent. Theory **9** (2005), 297–326, DOI 10.1090/S1088-4165-05-00277-3. MR2133762

[Schr11] Benjamin Schraen, *Représentations localement analytiques de* $\mathrm{GL}_3(\mathbb{Q}_p)$ (French, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 1, 43–145, DOI 10.24033/asens.2140. MR2760195

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

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